Extremal Processes in Branching Brownian Motion and Friends

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Abstract

In recent years there has been increasing interest in understanding the extremal behaviour of strongly correlated systems.

The main questions can be put as follows. Is there a rescaling of the maximum of the the process such that the rescaled maximum converges to a non-trivial limit? And if the answer is yes, what limiting object do we obtain if we rescale the whole process in that way?

In classical extreme value theory the answers to these questions are given for independent, identically distributed random variables (see e.g. [51] and [66] for a review). We start the introduction by recalling the main results for independent, identically distributed random variables. Next, we turn to the special case of Gaussian distributed random variables. In terms of comparison this is the most relevant case for the rest of this thesis. We then move on to certain strongly correlated random variables.

To understand the extreme value statistics of correlated systems it is a natural object to consider branching Brownian motion. Already Bramson proved in [26] and [24] that the level of the maximum is different from the one for independent, identically normal distributed random variables. Much more recently, Arguin et al. [4] and Aïdékon et al. [1] obtained the full extremal process.

In this thesis branching Brownian motion is the key object. In Chapter 2 and Chapter 3 we study variable-speed branching Brownian motion. In this model each particle is a time-changed Brownian motion. In this way we allow for a richer class of possible correlation structures. In the *weak correlation regime* we prove the convergence of the rescaled maximum and of the extremal process.

In Chapter 4 we extend the result in [4] and [1]. We add an additional dimension in such a way that also the information on the genealogical structure is encoded. This construction can be seen as an analogue to the $2d$ discrete Gaussian free field (see Biskup and Louidor [13]).

Chapter 5 is devoted to the study of the partition function in the complex temperature branching Brownian motion model in the glassy phase. This extends results by Madaule et al. [59]. The key ingredients are a precise understanding of the extremal process of branching Brownian motion and a clever way to compute moments of complex temperature partition functions.

In Chapter 6 we discuss the ageing phenomenon in the random energy model. This is a dynamical question concerning a certain Markov jump process in random environment. Again, a deep understanding of the extreme values of the energy landscape is needed.

Contents

CHAPTER 1

Introduction

1. Organization

Before starting the introduction let me explain its structure. As the title may suggest, a key word of this thesis is *extremal process* and its whole content is in one way or another linked to it. Hence, we start the exposition by the classical set up of extreme value theory for independent random variables along with some historical remarks. This is content of Section 2. At the end of this section we focus on the Gaussian case, in particular the *random energy model (REM)* which is in terms of comparison the most relevant for the remainder of this work.

In Section 3 we introduce a generalization of the REM where a certain correlation structure is added to the random variables, which leads to a set of random variables that are no longer independent.

Since a second main object in this work is *branching Brownian motion (BBM)*, we present some background and known results on its extremal behaviour in Section 4. The remaining sections of the introduction then give an overview of the original contributions of this thesis.

This is again divided into three main subjects. Sections 5 and 6. discuss results that are deeply related to extremal process of BBM and to the *generalized random energy model* and directly establish the convergence of certain extremal processes. Section 7 is devoted to the study of partition function in a certain model we called branching Brownian motion energy model. Finally, we discuss in Section 9 the ageing of the random energy model at its critical temperature.

2. Classical extreme value theory

Studying extremal events has been of great interest for a long time. For example, people were interested in studying the properties of unusual high floods of rivers or other very rare events. We start with a quick historical overview which is very much inspired by the one given by Emil Gumbel in [50]. The following simple question was already raised by Nicolas Bernoulli in 1709 in his essay *Specimina artis conjectandi, ad quaestiones juris applicatae* [11]. This example is taken from a book on the history of probability by Todhunter [73] p. 195.

Suppose that there are n men that will all die within k years. What is the probable duration of life of the last survivor?

Nicolas Bernoulli views this question the same as asking the following.

Take a line of length k with a fixed origin. Select n points at random on that line. What is the expected maximal distance to the origin?

Extremal events are by definition rare events. The question how long do we have to wait for a certain event was studied in Poisson's law. There, the main underlying question is "How often do rare events occur?". Interestingly it took roughly sixty years until in 1889 von Bortkiewicz [75] analysed the statistical meaning of this result for certain time series. He was interested in the number of suicides per year. The same author in [76] started analysing extremes as he studied the distance between the largest and smallest value of a realization of independent normal distributed random variables, which was then taken up by von Mises in [77]. Historically, the first articles concerned with the general theory for rare, respectively extremal events, mainly analysed the case of independent normal random variables. This was in some sense natural, since by the ordinary central limit theorem the occurrence of a normal event is approximately normal distributed. In this context also Tippetts work [72] should be mentioned. He included very precise statistical tables that were from a great practical relevance, so that the theoretically obtained results could actually be used. A more systematic study was initiated by Fréchet in [43] and then extended a year later by Fisher and Tippett in [42]. In these papers for the first time distribution free results were obtained (in particular cases). It turned out that distribution functions can be divided into different classes depending on the distribution of the maximal value. The first sufficient criteria that tell in which class a distribution function falls was given by von Mises in [78]. Gnedenko established in [48] necessary and sufficient conditions for this to hold.

It should also be remarked that the first systematic notes on extreme value theory that include theoretical distribution free results as well as a collection of material how they should be applied was written by Gumbel [50].

We start our mathematical summary on extreme value theory from independent and identical distributed random variables.

2.1. Independent identically distributed random variables. Let $(X_i)_{i\geq 1}$ be independent identically distributed random variables with common distribution function

$$F(x) = \mathbb{P}(X_1 \leq x). \tag{2.1}$$

To study the distribution of the maximal value of $X_1, \dots, X_n$ we set

$$M_n \equiv \max_{1\leq i\leq n} X_i, \tag{2.2}$$

which is itself a random variable depending on the random variables $X_1, \dots, X_n$. Our first aim is to analyse $\mathbb{P}(M_n \leq x)$ for n large and some $x \in \mathbb{R}$. Using that the X_i's are independent and identically distributed, we can rewrite this probability in the following way:

$$\begin{aligned} \mathbb{P}(M_n \leq x) &= \mathbb{P}\left(\text{For all } i \leq n : X_i \leq x\right) \\ &= \prod_{i=1}^{n} \mathbb{P}(X_i \leq x) \\ &= (F(x))^n. \end{aligned} \tag{2.3}$$

Looking at the quantity in (2.3), we observe that it always converges to either zero, if $F(x) < 1$ or to one, if $F(x) = 1$, as n tends to infinity. Hence, we would always observe a trivial limiting behaviour. The question is, whether this can be repaired, i.e. whether there are rescaling sequences a_n and b_n such that

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \to G(x), \quad \text{as } n \uparrow \infty, \tag{2.4}$$

where $G(x)$ is a non-trivial distribution function (meaning not zero or one for all values of $x \in \mathbb{R}$). This question is deeply related to the tail behaviour of the distribution function F. Performing similar manipulations as in (2.3) we have

$$\begin{aligned} \mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) &= \mathbb{P}\left(M_n \leq a_n x + b_n\right) \\ &= \left(F(a_n x + b_n)\right)^n . \end{aligned} \tag{2.5}$$

Hence, we can reformulate our question in the following way : Given a distribution function F, can we always find rescalings a_n and b_n such that $(F(a_n x + b_n))^n$ converges to a non-trivial limit $G(x)$ as n tends to infinity? And if yes, what are the possible limiting distribution functions $G(x)$? A remarkable theorem that answers both questions is due to Fréchet [43], Fisher and Tippett [42], and, in its most general form, to Gnedenko [48].

THEOREM 2.1. *Let $(X_i)_{i\in\mathbb{N}}$ be independent, identically distributed random variables. Then there always exists a rescaling such that*

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \to G(x), \quad \textit{as } n \uparrow \infty, \tag{2.6}$$

for some nontrivial distribution function. Moreover, G belongs to one of the following three types

(i) **Gumbel-distribution:** $G(x) = e^{-e^{-x}}$.

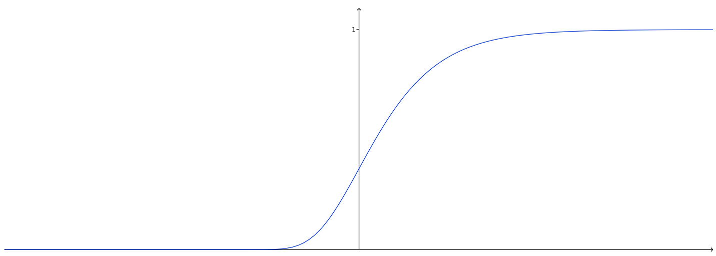


FIGURE 1. Gumbel distribution function

(ii) **Fréchet-distribution:** *For some* $\alpha > 0$, $G(x) = \begin{cases} 0 & \textit{if } x \leq 0 \\ e^{-x^{-\alpha}} & \textit{if } x > 0 \end{cases}$.

(iii) **Weibull-distribution:***For some* $\alpha > 0$, $G(x) = \begin{cases} e^{-(-x)^{-\alpha}} & \textit{if } x < 0 \\ 0 & \textit{if } x \geq 0 \end{cases}$.

The three distributions in (i), (ii) and (iii) are called *extremal type distributions*. Of course there is also a theory how to determine in which universality class particularly distributed $(X_i)_{i\in\mathbb{N}}$ belong. This can be phrased in terms of the tail behaviour of the distribution function. Then a distribution function is said to be in the *domain of attraction* of a Gumbel-/Fréchet-/Weibull-distribution. For a detailed study we refer to [66, 58]. For normal distributed random variables, we compute the limiting distribution of the maximum precisely in Section 2.2.

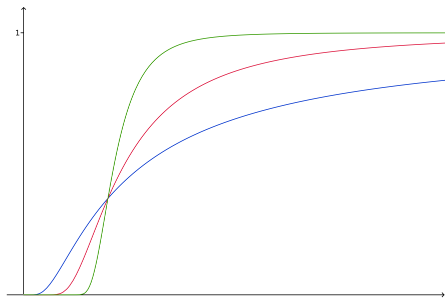


FIGURE 2. Fréechet distribution function. Blue: $\alpha = 1$, Red: $\alpha = 2$, Green: $\alpha = 5$

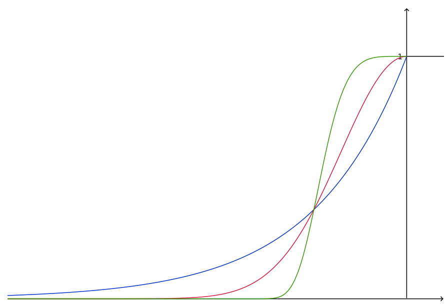


FIGURE 3. Weibull distribution function. Blue:$\alpha = 1$, Red: $\alpha = 2$, Green: $\alpha = 5$

Now, we extend the set of questions. We are interested in the following one. What happens if we not only rescale the maximum, but all the values of the X_i's in exactly the same way as we had to rescale the maximum? Or stated in an easier way.

How are the values close to the maximal value distributed?

To study this question we consider the set of points

$$\left(\frac{X_i - b_n}{a_n}\right)_{1 \le i \le n}, \tag{2.7}$$

respectively, more formally speaking, the point process $\mathcal{E}_{X_n}$, called *extremal process*, that is defined as

$$\mathcal{E}_{X,n} \equiv \sum_{i=1}^{n} \delta\left(\frac{X_i - b_n}{a_n}\right). \tag{2.8}$$

Does this point process converges as $n \uparrow \infty$ and if so, can we determine the limiting point process explicitly? In the case of identically distributed random variables, the answer to both questions is affirmative. We refer to [66], Corollary 4.19.

THEOREM 2.2. *Let $(X_i)_{i\in\mathbb{N}}$ be independent, identically distributed random variables and a_n and b_n as in Theorem 2.1. Then $\mathcal{E}_{X,n}$ converges weakly to a Poisson point process (PPP) whose intensity measure depends on its extremal type distribution. In particular:*

(i) *If (2.6) holds with G = Gumbel-distribution , then $\mathcal{E}_{X,n}$ converges weakly to a PPP* $(e^{-x}dx)$ *in* $M_p((-\infty,\infty])$[1] .
(ii) *If (2.6) holds with G = Fréchet-distribution, then $\mathcal{E}_{X,n}$ converges weakly to a PPP* $(x^{-\alpha}\mathbb{1}_{x>0}dx)$ *in* $M_p((0,\infty])$.
(iii) *If (2.6) holds with G = Weibull-distribution, then $\mathcal{E}_{X,n}$ converges weakly to a PPP* $((-x)^{-\alpha}\mathbb{1}_{x<0}dx)$ *in* $M_p((-\infty,0])$.

Hence the overall shape of the extremal process is always the same, it is a Poisson point process with a certain intensity.

Since this thesis is mainly concerned with the study of strongly correlated Gaussian processes the reference setting for us, is the case of independent identically distributed Gaussian random variables.

2.2. Gaussian random variables. Let $(X_i)_{i\in\mathbb{N}}$ be independent $\mathcal{N}(0,1)$ distributed random variables. In this particular case we answer similar questions as in Section2.1, in a more precise way.

(1) How do we have to rescale the maximum M_n to obtain a non-trivial limit?
(2) What is the limiting distribution function?
(3) How are the values close to the maximum distributed?, respectively, what is the limiting extremal process?

As we saw in (2.5) we have to study the tail behaviour of the distribution function. Since the X_i's are standard Gaussian random variables their distribution function Φ is given by

$$\Phi(x) = \int_{-\infty}^{x} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}. \tag{2.9}$$

A standard tail estimate gives (see e. g. Eq. (1.2.11) in [18]), that for x large,

$$\frac{1}{x\sqrt{2\pi}}e^{-x^2/2}(1-2x^{-2}) \le 1-\Phi(x) \le \frac{1}{x\sqrt{2\pi}}e^{-x^2/2}. \tag{2.10}$$

Using this crucial estimate one can prove the following theorem (see e. g. Section 4.2.2 in [18]).

THEOREM 2.3. *Let $(X_i)_{i\in\mathbb{N}}$ be independent standard Gaussian random variables. Let*

$$b_n = \sqrt{2\log n} - \frac{\log\log n + \log 4\pi}{2\sqrt{2\log n}} \tag{2.11}$$

and

$$a_n = \sqrt{2\log n}. \tag{2.12}$$

Then:

(i) *The rescaled maximum converges to a Gumbel distribution,*

$$\lim_{n\uparrow\infty}\mathbb{P}\left(\frac{M_n-b_n}{a_n}\le x\right) = e^{-e^{-x}}. \tag{2.13}$$

(ii) *The limiting extremal process is a Poisson point process (PPP) with intensity* $e^{-x}dx$,

$$\sum_{i=1}^{n}\delta\left(\frac{X_i-b_n}{a_n}\right) \to PPP(e^{-x}dx). \tag{2.14}$$

[1] $M_p(E)$ is the set of all point measures defined on E

Since our motivation came from spin glass models, let us now build a bridge and relate the setting in Theorem 2.3 to the simplest spin glass model, the *random energy model (REM)*. The REM was first introduced by Derrida in [31] and [32]. It is a stochastic process with state space is the N dimensional hypercube $\Sigma_N = \{-1,1\}^N$. To each $s \in \Sigma_N$ we associate an independent $\mathcal{N}(0,N)$ distributed random variable. This is in fact closely related to the setting in Theorem 2.3. Set

$$\begin{aligned} n &= 2^N \quad \text{(cardinality of } S_N) \\ Y_i &\equiv \sqrt{N} X_i \quad \text{where } Y_i \text{ is the r.v. attached to the i'th element of } \Sigma_N. \end{aligned} \tag{2.15}$$

Let us now consider the rescaling from Theorem 2.3.

$$\frac{X_i - b_n}{a_n} = \frac{Y_i - \sqrt{N} b_n}{\sqrt{N} a_n} = \frac{Y_i - \sqrt{N} b_{2^n}}{\sqrt{N} a_{2^n}}. \tag{2.16}$$

Setting

$$\begin{aligned} b_n^{\text{REM}} &\equiv \sqrt{N} b_{2^N} = \sqrt{2\log 2} N - \frac{\log(N\log 2) + \log 4\pi}{2\sqrt{2\log 2}} \\ a_n^{\text{REM}} &\equiv \sqrt{N} a_{2^N} = \sqrt{2\log 2} N \end{aligned} \tag{2.17}$$

we have

COROLLARY 2.4. *In the random energy model with rescaling as in (2.17)*

(i) *the rescaled maximum converges to a Gumbel distribution.*

$$\lim_{n\uparrow infty} \mathbb{P}\left(\frac{M_n - b_n^{REM}}{a_n^{REM}} \leq x\right) = e^{-e^x}. \tag{2.18}$$

(ii) *the limiting extremal process is a Poisson point process(PPP) with intensity* $e^{-x}dx$.

$$\sum_{i=1}^{n} \delta\left(\frac{X_i - b_n^{REM}}{a_n^{REM}}\right) \to PPP(e^{-x}dx). \tag{2.19}$$

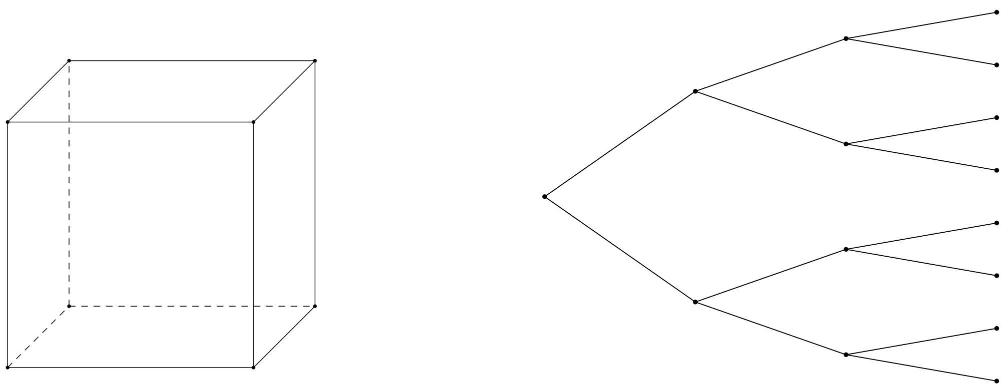
2.3. The REM on a tree. Since branching Brownian motion is a Gaussian process labelled by a tree that encodes the underlying branching structure. Let us start to think about a very easy model that we can label by a tree, namely a version of the random energy model.

Going back to the last section we remember that the random energy model was defined on the hypercube Σ_N. We can identify Σ_N with a binary tree of depth N.

We obtain the random energy model if we attach to each leave of this binary tree an independent $\mathcal{N}(0,N)$-distributed random variable. We can associate a distance to this binary tree in the following way. If we take two leaves i and j we count the number of branches they share starting from the root. We could also describe that by adding a time dimension to the tree. At time 0 we start with one particle that splits instantaneously into two. After time 1 each particle splits again into two particles and so on. For example, after time $N = 3$ we obtain the tree in Figure 4. The distance $d(i,j)$ between two leaves i and j (which correspond to two particles) in the tree is than given by

$$d(i,j) = \text{time of the last common ancestor of the particles } i \text{ and } j \tag{2.20}$$

Another canonical tree in the study of branching processes is the tree generated by a Galton-Watson process. Let us start by recalling what a *Galton-Watson process* is. We



(a) Hypercube S_N for $N = 3$ (b) Binary tree of depth $N = 3$

FIGURE 4. Correspondence between the elements of a binary tree of depth N and the N dimensional hypercube.

start with one individual at time zero. After an exponential time T with mean 1 it dies and gives birth to k offsprings with probability p_k. We only want to consider supercritical Galton-Watson processes. Hence, we assume for simplicity that $p_0 = 0(\sum_{k=1}^{\infty} p_k = 1)$. Each of the new individuals is subject to the same splitting rule.

At time t there are $n(t)$ individuals, $(i_k(t), k \leq n(t))$, alive. We can define the genealogical distance of two individuals $i_k(t)$ and $i_l(t)$ by

$$d(i_k(t), i_l(t)) = \text{time the two individuals } i_k \text{ and } i_l \text{ had a common ancestor for the last time.} \tag{2.21}$$

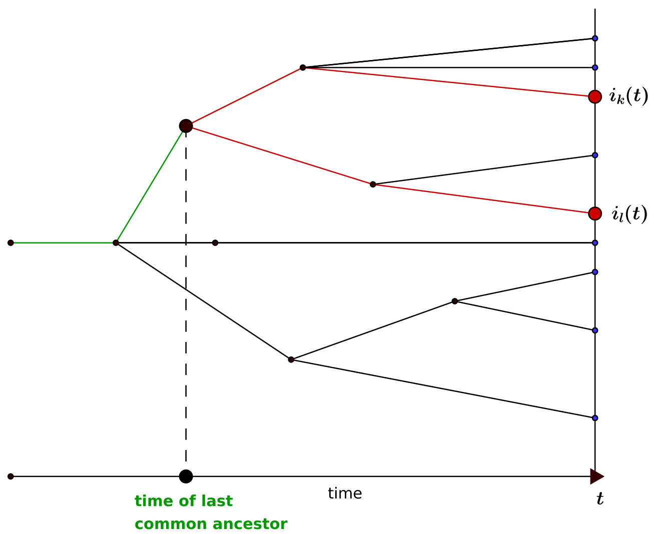


FIGURE 5. Distance of two individuals in the Galton Watson tree.

This leads to a tree- the so called *Galton-Watson tree*. Let us now combine the Galton-Watson tree with the random energy model. We associate to each leave of the Galton-Watson tree at time t independent $\mathcal{N}(0,t)$-distributed random variables,

$$(Z_i(t))_{1\leq i\leq n(t)}. \tag{2.22}$$

With our choice of the parameters of the offspring distribution, we have that

$$\mathbb{E}n(t) = e^t. \tag{2.23}$$

Moreover, it follows that (see e.g. Chapter 10 in [6])

$$\frac{n(t)}{\mathbb{E}n(t)} \to \mathfrak{C} \quad \text{as } t\uparrow\infty, \tag{2.24}$$

where $\mathfrak{C}$ is some random variable with expectation one. Using (2.24) one can prove the following (see also Section 4.6 in [18]).

THEOREM 2.5. *Taking the random function*

$$u_t(x) = t\sqrt{2} - \frac{\ln(t)}{2\sqrt{2}} + x + \frac{\log(n(t)/\mathbb{E}n(t))}{\sqrt{2}} \tag{2.25}$$

we have

(i) $\mathbb{P}\left(M(t)\leq u_t(x)\right) \to \exp\left(-\frac{1}{4\pi}e^{-\sqrt{2}x}\right)$

(ii) $\sum_{k\leq n(t)} \delta_{u_t^{-1}(x_k(t))} \to$ *PPP*$(\frac{1}{4\pi}e^{-\sqrt{2}x}dx)$.

We could also reformulate the results in Theorem 2.5 using the following $\widetilde{u}_t(x)$.

$$\widetilde{u}_t(x) = t\sqrt{2} - \frac{\ln(t)}{2\sqrt{2}} + x, \tag{2.26}$$

instead of the $u_t(x)$ as defined in (2.25). Then we have, that

(i) $\mathbb{P}\left(M(t)\leq \widetilde{u}_t(x)\right) \to \mathbb{E}\left(\exp\left(-\frac{1}{4\pi}\mathfrak{C}e^{-\sqrt{2}x}\right)\right)$,

(ii) $\sum_{k\leq n(t)} \delta_{\widetilde{u}_t^{-1}(x_k(t))} \to \mathbb{E}\left(\text{PPP}\left(\frac{1}{4\pi}\mathfrak{C}e^{-\sqrt{2}x}dx\right)\right)$,

where (i) and (ii) the expectation on the right hand side is taken over the random variable $\mathfrak{C}$. Hence, we have to options to encode the randomness coming from the underlying Galton-Watson process. Either we choose a random shift $u_t(x)$ of the maximum or we take a deterministic shift but obtain a randomly shifted Gumbel distribution in the limit (resp. a random intensity in the Poisson point process). A similar phenomenon appears later when we study the extremal process of Branching brownian motion.

3. The Generalized Random Energy Model

We turn to models where the random variables are no longer independent but have a certain correlation structure. The following so-called generalised random energy model (GREM) was proposed by Derrida in [33] as a generalization of the random energy model, discussed in the previous subsection. Since a Gaussian process is fully characterized by its covariance function. From now on, we use this description. Let us briefly go back to the REM on a binary tree. There, the covariance is given by

$$\text{Cov}\left(x_i^N, x_j^N\right) = \begin{cases} N & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}. \tag{3.1}$$

In particular, the covariance does not depend on the distance of i and j in the binary tree (if they are not the same). This changes in the generalized random energy model (GREM), where

$$\text{Cov}\left(x_i^N, x_j^N\right) = \begin{cases} N & \text{if } i = j \\ N \cdot A\left(\frac{d(i,j)}{N}\right) & \text{if } i \neq j \end{cases}, \tag{3.2}$$

where $d(i,j)$ is defined in (2.20) and $A : [0,1] \to [0,1]$ has the following properties:

(i) $A(0) = 0$ and $A(1) = 1$.

(ii) A is an increasing stepfunction.

If A also satisfies

(iii) $A < x$ for all $x \in [0,1]$.

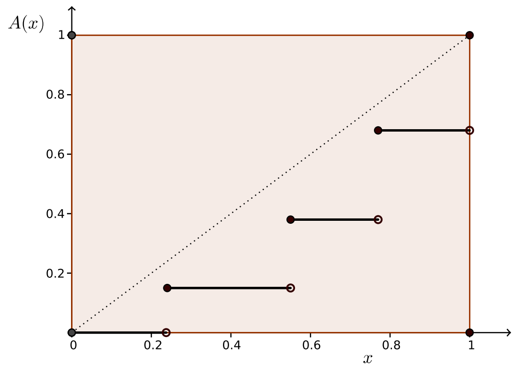


FIGURE 6. Example for a function $A : [0,1 \to [0,1]$ that satisfies $(i), (ii)$ and (iii). In particular, it is a step function with $A(x) < x$ for $x \in (0,1)$.

Bovier and Kurkova in [22] proved the convergence of the properly rescaled maximum to a Gumbel distribution and established the convergence of the extremal process to a Poisson point process. To omit too much notation, we state their results in a rather informal way.

THEOREM 3.1. *In the GREM where A satisfies $(i), (ii)$ and (iii) the following is true.*

1. *The level of the maximum coincides with the one in the REM.*
2. *The maximum rescaled exactly as in Corollary 2.4 converges to a Gumbel distribution.*
3. *The extremal process converges to the same Poisson point process as in 2.4.*

In the more general case where $A(x)$ is allowed to be larger than x the picture changes. In [22] the following was shown

THEOREM 3.2. *In the GREM where A satisfies $(i), (ii)$ but $A(x) > x$ for some $x \in (0,1)$ the following is true.*

1. *The first order of the maximum depends on the concave hull of A, which we call $\widehat{A}$ and its right derivative by $\left(\widehat{A}\right)'$, more precisely*

$$\frac{M_n}{\sqrt{2\log(2)N}\int_0^1 \sqrt{\left(\widehat{A}\right)'(x)}} \to 1, \tag{3.3}$$

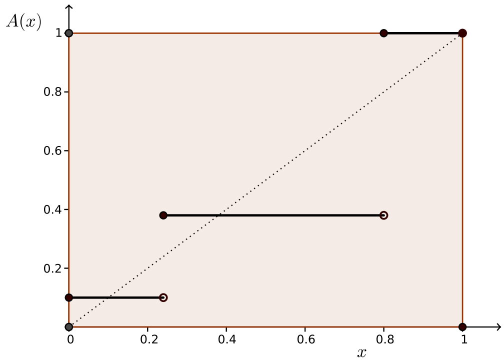


FIGURE 7. Example for a function $A : [0, 1 \to [0, 1]$ that satisfies (i), (ii), but there exists $x \in (0, 1)$ such that $A(x) > x$.

as $n \uparrow \infty$ in probability.

2. *The maximum can be rescaled such that it converges. The resulting limit to a shifted Gumbel distribution.*
3. *The properly rescaled extremal process converges to a cascade of Poisson point processes. .*

A cascade of Poisson point process is a concatenation of different Poisson point processes. So we generate the first Poisson point process. And then for each point we generate independent copies of the second and at these points to this point and so on. Mathematically this can be made more precise , we refer to Ruelle [68]. Let us first make some comments on Theorems 3.1 and 3.2. We want to point out a few things about the genealogical structure of extremal particles in the two settings. Consider two extremal particles at time N. Then we follow their paths backward in time and ask the question:

When will the two trajectories meet (with high probability)?

Interestingly, the answer heavily depends on the properties of A.

- In the setting of Theorem 3.1 the answer is that they will meet at time zero. So their paths split directly and are independent.
- Turning to the setting of Theorem 3.2 this is no longer true. With positive probability the paths can meet at a Nh, for each discontinuity point h of the concave hull $\widehat{A}$ with $A(h) > h$. This leads to a ”restart” of a extremal process in these positions and causes that the extremal process has a concatenated structure. At each time Nh an extremal particle has already to be maximal and then all extremal particles produce extremal particles at a later time.

This model can be further extended to the continuous random energy model (CREM), where more general functions A are allowed. As already seen from Theorem 3.1 and 3.2, it is expected that the extremal process depends heavily on the properties of A. Bovier and Kurkova in [23] showed that the leading order still depends on the concave hull of A but they did not obtain further information on the subleading orders of the maximum and the properties of the extremal process. For a more detailed description see [14]. It turns out to be useful to first study basically the same model but on a slightly different tree, namely the Galton-Watson tree, where certain technical tools are available.

This is closely related to a stochastic process called *branching Brownian motion (BBM)*. Hence, we first introduce BBM. And then come back to the its relation to the CREM.

4. Branching Brownian motion

Branching Brownian motion was already introduced in [65], [71] in the late 1950's and early 1960's. Seminal contributions were made by McKean [64], Bramson [26, 24], Lalley and Selke [57] and Chauvin and Rouault [28, 29] in the 1970's and 1980's on its connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation and its rescaled maximum. The later is even more classical and was already studied by Kolmogorov, Petrovsky and Piscounov [56] and Fisher [41] in 1937.

In the last years there has been a revival of the interest in branching Brownian motion. Hence, a number of very nice lecture notes or review articles have been written. There are nice lecture notes by Shi [69, 70] focusing on spinal decomposition. Concerning the branching random walk and its maximal displacement there are lecture notes by Zeitouni [79]. The extensive lecture notes by Bovier [18] cover most of the material of this introduction. There is the following intuitive construction of branching Brownian motion.

1. Start a standard Brownian motion X at $X(0) = 0$.
2. After an exp(1) distributed time T the Brownian motion splits into k particles with probability p_k that we choose such that,

$$\sum_{k=1}^{\infty} p_k = 1, \quad \sum_{k=1}^{\infty} k p_k = 2, \quad \sum_{k=1}^{\infty} k(k-1) p_k \equiv K < \infty. \tag{4.1}$$

3. Each of the new particles moves according to independent Brownian motions starting in $X(T)$.
4. Each particle is then subject to the same splitting rule.

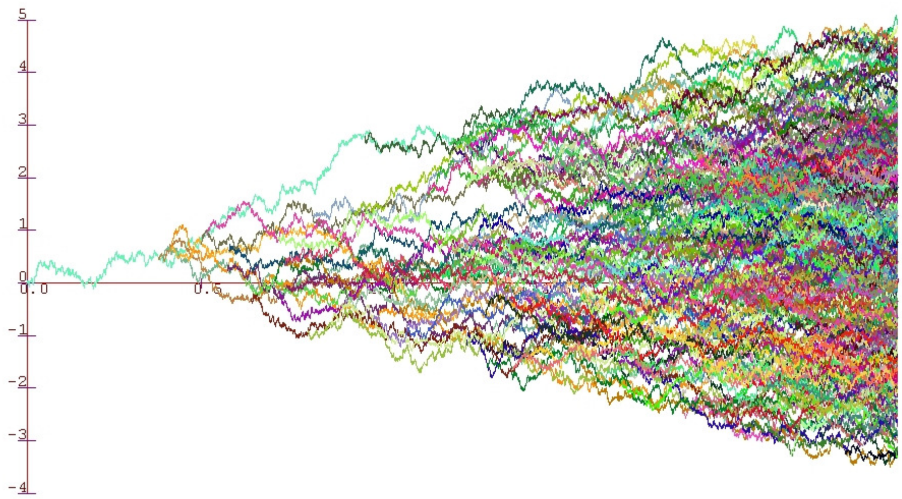


FIGURE 8. Picture of branching Brownian motion (taken from Matt Roberts).

We denote the number of particles of BBM after time t by $n(t)$ and the set of particles at that time by

$$\{x_k(t), k \leq n(t)\}. \tag{4.2}$$

Alternatively, we can view branching Brownian motion as a Gaussian process, indexed by the leaves of a Galton-Watson process. Let $d(\cdot, \cdot)$ as in (2.21). Then, for a given realization of the Galton-Watson process, the collection $\{x_k(t), k \leq n(t)\}$ is the Gaussian process with mean zero and Covariance

$$\mathbb{E}\left(x_k(t)x_l(t)\right) = d(i_l(t), i_k(t)) \tag{4.3}$$

Thus BBM is the analog to the CREM introduced at the end of Section 3 with $A(x) = x$, when one replaces the binary tree by a Galton-Watson tree.

4.1. BBM and the F-KPP equation. A crucial tool in the study of extremal particles of BBM is the deep connection between BBM and a certain partial differential equation, the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation:

$$\partial_t v(x,t) = \frac{1}{2}\partial_x^2 v(x,t) + F(v), \tag{4.4}$$

where (in our setting)

$$F(v) \equiv (1-v) - \sum_{k=1}^{\infty} p_k (1-v)v^k. \tag{4.5}$$

The F-KPP equation is a well understood reaction-diffusion equation. It was first studied by Fisher in [41] and Kolmogorov, Petrovsky and Piscounov in [56]. Fischer used this equation in [41] to model the evolution of biological populations. It accounts for: birth v, death:$-v^2$ and diffusive migration $\partial_x^2 v$. The fundamental link between BBM and the F-KPP equation (4.4) is generally attributed to McKean [64]. However, it already appeared in Skorohod [71] and Ikeda, Nagasawa, and Watanabe [53, 54, 55]. They observed that expectations of certain functionals of BBM particles solve the F-KPP equation.

THEOREM 4.1. *Let $f : \mathbb{R} \to [0,1]$ and $\{x_k(t) : k \leq n(t)\}$ BBM. Set, for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,*

$$u(t,x) = \mathbb{E}\left[\prod_{k=1}^{n(t)} f(x - x_k(t))\right]$$

Then $v \equiv 1-u$ is the solution of the F-KPP equation with initial condition $v(0,x) = 1 - f(x)$.

This lemma has a lot of consequences. Let us first remark that the implications of Theorem 4.1 on the study of the maximum.

REMARK. Two main implications of Theorem 4.1 are the following.

(i) Similar to (2.3) we have

$$\mathbb{P}\left(M_n \leq x\right) = \mathbb{P}\left(\text{For all } k \leq n(t) : x_k(t) \leq x\right) \tag{4.6}$$

$$= \mathbb{E}\left(\prod_{k=1}^{n(t)} \mathbb{1}_{\{x_k(t) \leq x\}}\right). \tag{4.7}$$

Hence, $1 - u(t,x) = 1 - \mathbb{P}\left(M_n \leq x\right)$ solves the F-KPP equation.

(ii) Theorem 4.1 is also applicable to Laplace functionals which encode all information on the extremal process. We come back to that later.

Therefore, understanding solutions of the F-KPP equation is a key tool to study the behaviour of extremal particles of BBM. In the following we state some basic results on the F-KPP equation. A key feature of the F-KPP equation is that it admits travelling wave solutions.. A travelling wave solution u to the F-KPP equation (4.4) with speed λ is a solution such that

$$\frac{d}{dt}u(t, x + \lambda t) = 0. \tag{4.8}$$

A computation shows that this implies $u(t, x + \lambda t) = w_\lambda(x)$, where $w_\lambda(x)$ is the solution of

$$\frac{1}{2}\partial_x^2 w_\lambda(x) + \lambda\partial_x w_\lambda(x) + F(w_\lambda(x)) = 0 \tag{4.9}$$

We are looking for solutions that decay to zero at plus infinity. Therefore, for large positive x, w_λ must be close to the linearised equation

$$\frac{1}{2}\partial_x^2 w_\lambda(x) + \lambda\partial_x w_\lambda(x) + w_\lambda(x) = 0 \tag{4.10}$$

Now one has to distinguish two cases $\lambda \neq \sqrt{2}$ and $\lambda = \sqrt{2}$.

(i) $\lambda \neq \sqrt{2}$: There are two linearly independent solutions to (4.9) of the form

$$e^{-b_\pm x} \quad \text{with } b_\pm = \lambda \pm \sqrt{\lambda^2 - 2}. \tag{4.11}$$

(ii) $\lambda = \sqrt{2}$: There are two linearly independent solutions to (4.9) of the form

$$e^{-\sqrt{2}x} \quad \text{and} \quad xe^{-\sqrt{2}x}. \tag{4.12}$$

Using a phase space analysis Kolmogorov et. al. in [56] and Uchiyama in [74] proved, that in both cases the heavier tailed solution describes the correct asymptotics of the travelling wave solution. Moreover, they proved that travelling wave solution are unique up to translations.

THEOREM 4.2. *Let F satisfy the following assumptions.*

(i) $F \in C^1([0,1])$*: F is a continuously differentiable function from* $[0,1]$ *to* $\mathbb{R}$.
(ii) $F(0) = F(1) = 0, F(u) > 0, \forall u \in (0,1)$ *and* $F'(0) = 1, F'(u) \leq 1, \forall u \in [0,1]$.
(iii) $1 - F'(u) = O(u^\rho), \rho < 1$.

Then, for $\lambda \geq \sqrt{2}$, (4.9) has a unique solution up to translation, which satisfies

$$0 < w_\lambda(x) < 1, \quad w_\lambda(x) \to 0 \text{ as } x \to \infty, \quad w_\lambda(x) \to 1, \text{ as } x \to -\infty. \tag{4.13}$$

We are interested in the properties of extremal particles in BBM. We will see that that the relevant case for the study of BBM are the travelling waves with speed $\lambda = \sqrt{2}$. This might have been guessed from the fact that $u_t(x)/t = \sqrt{2} + o(1)$ in the i.i.d. setting in (2.25). The convergence to the travelling was completely analysed by Bramson in [24]. The following is a slightly specialized version of Theorem A and B in [24]. The formulation below is taken from [17].

THEOREM 4.3. *Let v be a solution to the F-KPP equation (4.4) satisfying Assumption $(i)-(iii)$ from Theorem 4.2 and $0 \leq v(0,x) \leq 1$. Then there exists a function $m(t)$ such that*

$$v(t, x + m(t)) \to \omega(x), \tag{4.14}$$

uniformly in x, where ω is a solution (4.9), as $t \uparrow \infty$, if and only if

(i) *for some $h > 0$, $\limsup_{t\to\infty} \frac{1}{t}\ln\int_t^{t(1+h)} v(0,y)dy \leq -\sqrt{2}$, and*
(ii) *for some $\nu > 0$, $M > 0$, $N > 0$, $\int_x^{x+N} v(0,y)dy > \nu$ for all $x \leq -M$.*

Moreover, if $\lim_{x\to\infty} e^{bx}v(0,x) = 0$ for some $b > \sqrt{2}$, then one may choose

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t. \tag{4.15}$$

To emphasise the importance Theorem 4.3 observe the following. Taking initial value

$$v(0,x) = 1 - \mathbb{1}_{0\le x} = \mathbb{1}_{0>x} \tag{4.16}$$

satisfies the Conditions in Theorem 4.3. Hence, we have the following Corollary.

COROLLARY 4.4. *Let $M_n = \max_{k\le n(t)} x_k(t)$. Then, with $m(t)$ as in (4.15), we have*

$$\mathbb{P}\left(M_n - m(t) > x\right) = 1 - \mathbb{P}\left(M_n - m(t) \le x\right) \to 1 - w_{\sqrt{2}}(x), \tag{4.17}$$

as $t \uparrow \infty$ and where $w_{\sqrt{2}}(x)$ is a solution to (4.9) with $\lambda = \sqrt{2}$.

Observe that the rescaling factor differs from the one in the i.i.d. case which is equal to

$$\sqrt{2}t - \frac{1}{2\sqrt{2}}\log t. \tag{4.18}$$

Therefore, we see that already on the level of the maximum BBM differs from the i.i.d. case. This tells us that the particles start to feel the correlations and do not behave as in the independent case. This effect is so weak that it is not seen in the linear term (which still is $\sqrt{2}t$) but in the logarithmic correction.

4.2. The derivative martingale. One can also represent the travelling wave $w_{\sqrt{2}}(x)$ in a stochastic way. Namely, we can write the travelling wave as an expectation over a randomly shifted Gumbel distribution. The following theorem is due to Lalley and Selke in [57].

THEOREM 4.5. *Let $w_{\sqrt{2}}(x)$ be as before. Then*

$$w_{\sqrt{2}}(x) = \mathbb{E}\left(\exp\left(-CZe^{-\sqrt{2}x}\right)\right), \tag{4.19}$$

for some constant $C > 0$ and Z is a random variable. In particular, Z is the limit of the so-called derivative martingale $Z(t)$,

$$Z = \lim_{t\to\infty} Z(t) \equiv \lim_{t\to\infty} \sum_{k\le n(t)} (\sqrt{2} - x_k(t))e^{\sqrt{2}x_k(t)-2t}. \tag{4.20}$$

We remark that the limit of $Z(t)$ exists almost surely. Moreover, Z encodes certain informations about the early history of the process. Intuitively speaking, Z answers the following question:

How many particles are generated at the very beginning (and that evolve independent afterwards) that have a chance of being extremal at time t?

Hence, it is not surprising that it appears as a random shift in (4.19). If there is a higher number of candidates in the beginning, the maximal one will get higher. In contrast having less candidates reduces the maximal value.

Let us compare (4.19) with the analogue result in Theorem 2.5. In Theorem 2.5 also a random shift appeared, namely $\mathfrak{C}$, that was simply counting the number of individuals. The random variable Z additionally weights the position of all particles. We will see in Chapter 2 and Chapter 3 that also different shifts can occur that are again limits of certain martingales.

4.3. The extremal process of BBM. The question whether the properly rescaled maximum of BBM particles converges was, as already mentioned, answered by Bramson in [24] and a more probabilistic interpretation of the limiting law was given by Lalley and Selke in [57]. The question about the convergence of the extremal process stayed unanswered until it was more recently answered by Arguin, Bovier and Kistler in [4] and by Aïdékon, Berestycki, Brunet and Shi in [1]. See also [49] for a comparative review. There, it was shown that the limit of the extremal process exists and is given by

$$\lim_{t\uparrow\infty} \widetilde{\mathcal{E}}_t \equiv \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{x_k(t)-m(t)} = \widetilde{\mathcal{E}}, \tag{4.21}$$

exists in law, and $\widetilde{\mathcal{E}}$ is of the form

$$\widetilde{\mathcal{E}} = \sum_{k,j} \delta_{\eta_k+\Delta_j^{(k)}}, \tag{4.22}$$

where η_k is the k-th atom of a Poisson point process with random intensity measure $CZe^{-\sqrt{2}y}dy$, with C and Z as in (4.20), and $\Delta_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)}\tilde{x}_j(t)}, \tag{4.23}$$

where $\tilde{x}(t)$ is BBM conditioned on the event $\{\max_{j\leq n(t)}\tilde{x}_j(t) \geq \sqrt{2}t\}s$.

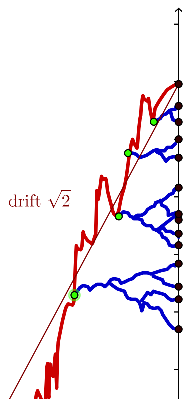


FIGURE 9. The extremal process consists of two elements: A Poisson point process representing a local maxima (in red) and independent copies of the cluster process "glued" to each Poisson point (in blue).

To understand this result it is helpful to think about the following question concerning the path an extremal particle at time t has taken.

Was such an extremal particle at time t among the leading particles for a long time or is it more likely that it was in the bulk of particles nearly all the time and was then just selected from a lot of particles at the very end?

The answer to this question was obtained in [3]. A leading particle at time t was, with high probability, of order $\sqrt{t}$ below the maximal particle at time s for $r < s < t - r$, where r is very small compared to t. In particular, it was located in regions where many particles can be found.

This implies a very natural second question.

> *If we select two extremal particles at time t (e.g. whose position is $> m(t) - d$ for some constant d) and follow their paths backward in time, when will these paths merge?*

This question has also been studied in [3]. Its answer is very convincing in view of the answer of the first question raised here. There are actually two cases. Either the paths merge more or less directly or they merge at the very end (cf. Figure 10).

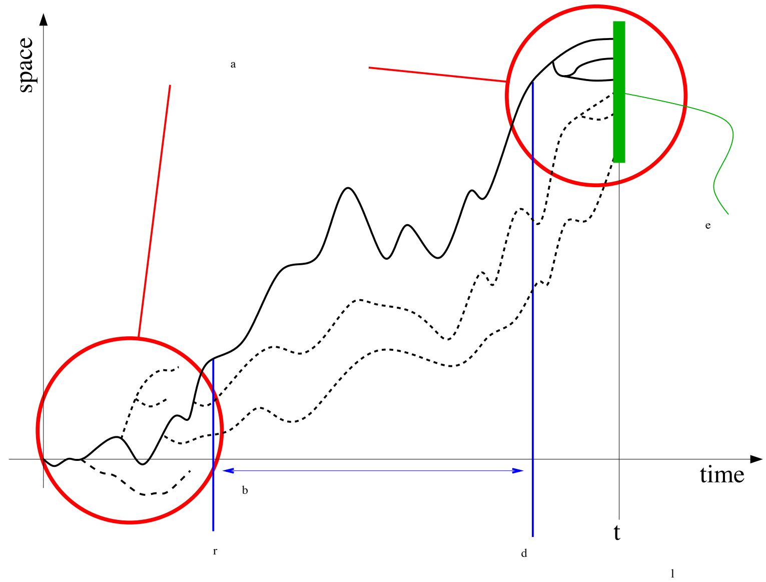


FIGURE 10. Illustrating the two cases when the paths of two extremal particles at time t merge (Picture by Nicola Kistler).

Now, let us think about the consequences for the structure of the limiting extremal process. In the first case they spend almost all their time together and so their positions are also very close at time t. Hence they belong to the same cluster (see Figure 9). In the second case they moved independently from each other almost all time and therefore also there positions at time t are basically independent. This means that they are both points of the Poisson point process, that already appeared when we discussed the extremal process of independent random variables.

Putting these observations together we obtain the structure described in (4.22).

5. Variable Speed Branching Brownian motion

In the remainder of this introduction, we give a brief outline of the original results of this thesis that are presented in detail in Chapters 2-6. This section is devoted to the study of variable speed BBM and discusses on a heuristic level the content of Chapter 2 (see [20]) and Chapter 3 (see [21]) of this thesis. Additionally, we comment on related articles that were written in the past years by other groups on very related issues. One should mention that instead of changing the Brownian movement one could also make according changes to the underlying Galton Watson process (e. g. speeding up or down the rate at which branchings occur). That leads to exactly the same model.

We want to point out that this section is not a formal and technically correct descriptions of the results obtained and their proofs. We refer the interested reader to Chapters 2 and 3 , respectively [20] and [21], for exact statements of the theorems and their proofs.

To allow a richer class of covariance functions we study variable speed branching Brownian motions. In view of the GREM and CREM that is a natural extension of BBM. It was first proposed by Derrida and Spohn in [34]. For a given realization of the Galton-Watson process $\{x_k^A(t), k \leq n(t)\}$ is the mean zero Gaussian process with covariance

$$\mathbb{E}\left(x_k^A(t)x_l^A(t)\right) = \Sigma^2(d(i_l(t), i_k(t)), \tag{5.1}$$

where

$$\Sigma^2(s) = tA\left(\frac{s}{t}\right). \tag{5.2}$$

The function $A : [0,1] \to [0,1]$ in (5.2) should be a non-decreasing function with $A(0) = 0$ and $A(1) = 1$. By (5.1) the movement of each particle is a time changed Brownian motion $B_{\Sigma^2(s)}$ and the particles are subject to the same splitting rule as in ordinary branching Brownian motion. We call $\Sigma^2(s)$ *speed function* and say that the Gaussian process defined in (5.1) is a variable speed BBM with speed function Σ^2.

Observe that the function A in (5.2) plays exactly the same role as in the CREM. Just the underlying tree is a Galton-Watson tree in contrast to the binary tree in the CREM.

5.1. Two linear segments. In Chapter 2 we study the case where A consists of two linear segments. Namely, we choose its derivative A' as

$$A'(s) = \begin{cases} \sigma_b^2 & 0 \leq s < b \\ \sigma_e^2 & b \leq s \leq 1 \end{cases}, \quad 0 < b \leq 1. \tag{5.3}$$

We normalise the total variance by assuming (see Figure 11)

$$\sigma_b^2 b + \sigma_e^2(1-b) = 1. \tag{5.4}$$

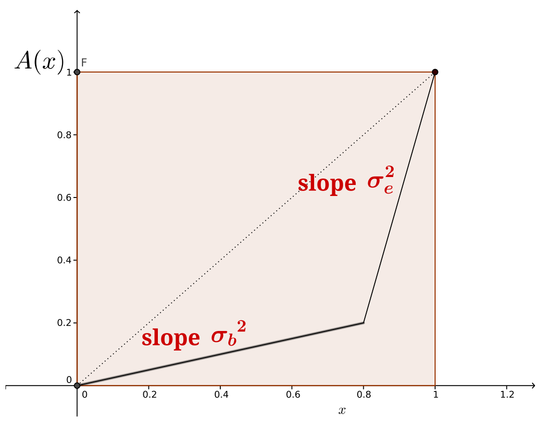


FIGURE 11. A is chosen to be a function that has slope σ_b until time b and then slope σ_e. The figure shows an example where $\sigma_b < \sigma_e$.

Improving results by Fang and Zeitouni in [39] on the maximal displacement, we prove the full convergence of the extremal process. We prove the following.

1) $\sigma_b < \sigma_e$: In the case where the first slope is strictly less than one the limit is again a decorated Poisson point process with a random intensity measure (as in (4.22)). The random intensity measure is a martingale limit (called *McKean martingale*) depending only on the first slope. The decoration process only depends on the second slope and has a similar form as (4.23).
2) $\sigma_b > \sigma_e$: In the case where the first slope is larger than one the limiting process is a cascade of extremal processes of ordinary BBM's.

The precise statement can be found in Chapter 2, Theorem 1.2 and Theorem 1.3. Let us discuss the difference between the behaviour of the particles in the two cases. To this end we describe the most likely path of a particle reaching the maximal height.

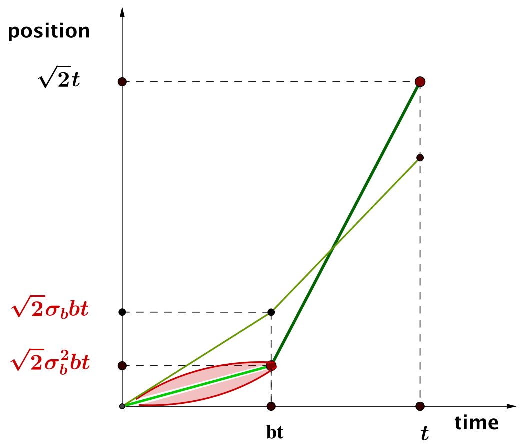


FIGURE 12. Path of a particle that is extremal at time t if $\sigma_b < \sigma_e$.

If $\sigma_b < \sigma_e$ the particle reaching the maximal height at time t is likely to be at position $\sqrt{2}\sigma_b^2 bt \pm O(t^{1/2})$ at the time of the speed change. Hence, it is much below the maximal particle at this moment (see Figure 12). This implies that the maximal particle is selected from exponentially many particles and only one of these particles will reach the maximal height at time t.

The situation is different when $\sigma_b > \sigma_e$. In this case an extremal particle at time t already has to belong to the leading particles at the time of the speed change (see Figure 13). Moreover, each particle, that is extremal at the time of the speed change, will produce a finite number of extremal particles at time t. This phenomenon leads to the concatenated structure of the extremal process.

5.2. The weak correlation regime. Our aim is to generalize these results to arbitrary time changes. Assume that the function in (5.2) has the following properties (see Figure 5.2):

(i) The slope σ_b of the tangent to A at 0 is smaller than 1.

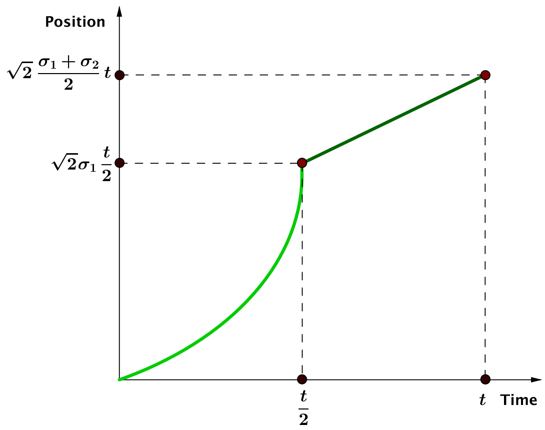


FIGURE 13. Path of a particle that is extremal at time t if $\sigma_b > \sigma_b$.

(ii) The slope σ_e of the tangent to A at 1 is larger than 1.
(iii) $A(x) < x$ for all $x \in (0,1)$ and $A(0) = 0$ and $A(1) = 1$.

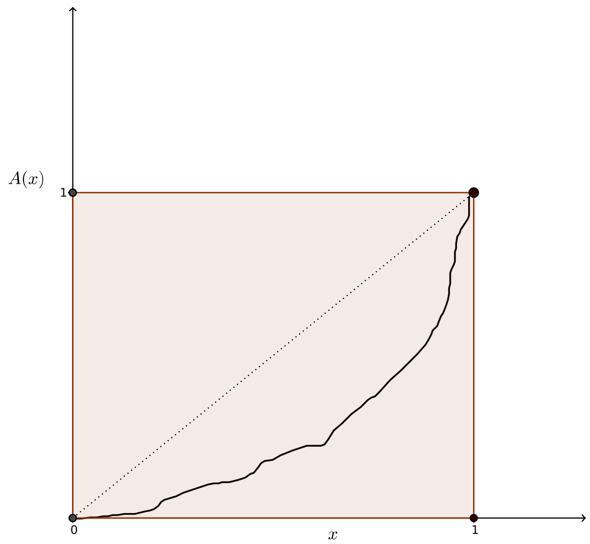


FIGURE 14. A function $A : [0,1] \to [0,1]$ satisfying the assumptions $(i) - (iii)$.

In Chapter 3, respectively [21], we prove the following result for any A that satisfies properties $(i) - (iii)$ (and some weak regularity in a neighbourhood of zero and one) on the extremal process.

The structure of the extremal process is 'universal', in the sense that it only depends on the slopes at 0 and 1. In particular, it is not affected by the form of the time change between 0 and 1.

The precise statement can be found in Chapter 3, Theorem 1.2. The precise requirements on A are conditions $(A1) - (A3)$ in Chapter 3. To this end, we want to compare the extremal behaviour of a variable speed BBM (with general function A) with the one where A consists of two linear segments.

We extend the standard Gaussian comparison techniques so that we can handle the case of variable speed BBM's. A key ingredient is again the correct localization of the path taken by a particle, that is extremal at time t. The idea is that the path over the time interval is always close to the function $s \mapsto \sqrt{2}tA(s/t)$. To understand this let us consider one single particle $x_k(t)$ for some $k \leq n(t)$. Its path is a time changed Brownian motion $B_{tA(s/t)}$. Hence,

$$\xi_k(s) \equiv x_k(s) - \frac{s}{t}x_k(t) \tag{5.5}$$

is a time changed Brownian bridge from zero to zero in time t. For such a Brownain bridge it is known (see e.g. Lemma 2.2 in [21] or Chapter 2 in [24]) that its deviation from zero is smaller than t^γ for $\gamma > \frac{1}{2}$. For k such that $x_k(t) \approx \sqrt{2}t$ that translates exactly into the desired property. The behaviour is very similar to the one shown in Figure 11 for a

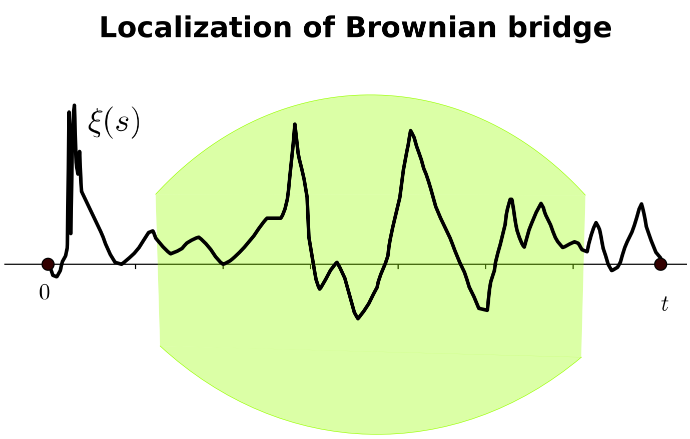


FIGURE 15. A Brownian bridge form zero to zero in time t. The green area represents the region in which the Brownian Bridge is with high probability.

variable speed BBM, where A consists of two linear segments. This leads to the claim that also the extremal processes should be the same.

Currently, we are trying to solve the complementary problem, when the speed function is strictly concave. There a more involved structure of extremal particles is expected. In [60] Maillard and Zeitouni identified the order of the maximum. They use Girsanov transform methods that (so far) do not lead to non trivial results on the extremal process. We want to give a more geometric description of the limiting extremal process and in particular of its genealogical structure. Preliminary heuristics suggest that there is an inhomogeneous Poisson process of times when branchings occur that lead to several offspring manage reaching extremal levels.

In the case of piecewise linear speed functions, we know that this happens exactly at the times of speed changes.

5.3. Related Articles. The study of variable speed BBM has been a quiet active field of research over the past years. In this subsection we want to mention some related works.

The first one to mention is the one by Fang and Zeitouni in [39] studying the level of the maximum in the case where A consists of two linear pieces for the case of branching random walk.

For the case of strictly concave functions A there has been an article by [40] where Fang and Zeitouni show that the order of the maximum has a correction of order $t^{1/3}$. In [60] the convergence of the maximum was established. Unfortunately, they do not obtain a very detailed description of the limiting law. Moreover, in generalized BRW setting Mallein obtained the correction term of order $t^{1/3}$ in [62].

Mallein and Milos considered in [63] the case of a branching random walk in a time-inhomogeneous random environment and studied the properties of the maximum in that case.

Another related article is the one by Arguin and Ouimet [5] who considered an analogue setting for a scale dependent two dimensional Gaussian free field, that falls in the same universality class as BBM. They compute the first order of the maximum and the the $\log$-number of high points.

6. Extended convergence of the extremal process of branching Brownian motion

In this section we give a summary of [19], which is part of this thesis (namely Chapter 4). The description of the extremal process of BBM in [4] and [1] is rather implicit and we describe a construction that allows a disentanglement of the extremal process. This is inspired by the analysis of the 2d discrete Gaussian free field [13], where the extended convergence theorem is natural due to an additional space dimension. One should mention that there has also been a different approach by Mallein in [61] who considered the order of the maximum for a $d-$dimensional BBM. This should, when studying the extremal process, also lead to a disentanglement of the different clusters. Let us focus on the construction given in [19] that is in some sense more intrinsic. We shortly comment on the way we choose the embedding and its meaning for the extremal particles.

To get additional structural information on the particles of BBM, we embed the underlying Galton-Watson tree into $\mathbb{R}_+$ in such a way that the genealogical structure of the tree is respected. Let us discuss an analogous labelling for the binary tree of depth N (see Figure 4). Such tree can be identified with the sequences of 0 and 1 of length N, namely the space $\{0,1\}^N$. We can interpret this in the following way: the upper child always receives a 1 and the lower one a 0. We transfer this idea to the the tree generated by a Galton-Watson process. Of course, this is continuous time tree such that we have to introduce zeros to keep track of the order in which branchings occur. The resulting embedding of the leaves $\{i_1(t),\dots,i_k(t)\}$ we call $\gamma(\cdot)$.

This leads to a two dimensional process encoding also the correlation structure of the BBM particles in the following sense. If we take two particles at time t and there embedded points in $\mathbb{R}_+$ are not close then they must have split after some time r much smaller than t. Whereas points that branched off very late are mapped to the same point (in the limit $t\uparrow\infty$). Using this information about the extended process, we prove in Chapter 4, respectively [19], the two-dimensional convergence of the extremal process.

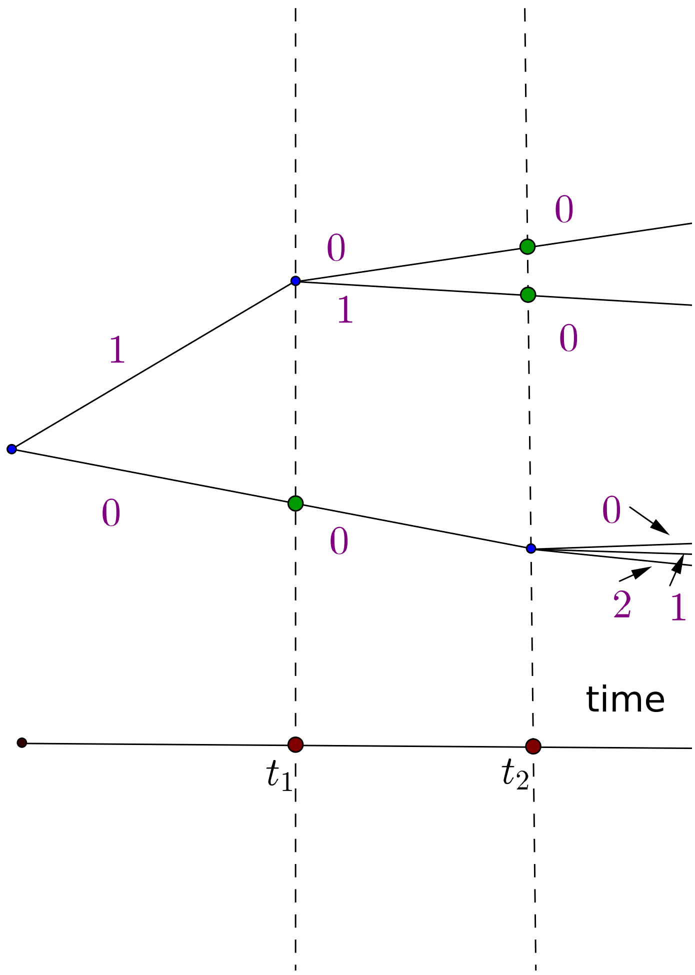


FIGURE 16. Example of a tree with labels of each branch. Number all children starting from zero each time a branching occurs; and add a zero label if the particle is not branching at the time. This leads to multi-labels representing the particles of the Galton-Watson tree.

The point process $\widetilde{\mathcal{E}}_t \equiv \sum_{k=1}^{n(t)} \delta_{(\gamma(i_k(t)),x_k(t)-m(t))} \to \widetilde{\mathcal{E}}$ on $\mathbb{R}_+ \times \mathbb{R}$, as $t \uparrow \infty$, where

$$\widetilde{\mathcal{E}} \equiv \sum_{i,j} \delta_{(q_i,p_i)+(0,\Delta_j^{(i)})}, \tag{6.1}$$

where $(q_i, p_i)_{i\in\mathbb{N}}$ are the atoms of a Cox process on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $Z(dv) \times Ce^{-\sqrt{2}x}dx$, where $Z(dv)$ is a random measure on $\mathbb{R}_+$ and $\Delta_j^{(i)}$ as in (4.22) the atoms of the cluster process.

The random measure $Z(dv)$ is deeply connected to the limit of the derivative martingale Z. Namely, $Z([0,\infty)) = Z$. The precise statement can be found in Chapter 4, Theorem 3.1. For a precise definition of the random measure $Z(dv)$ and its properties, we refer to Chapter 4, Lemma 3.2.

Knowing the genealogical structure of the extremal process of BBM (obtained in [3]) this leads to the following intuitive structure of the limiting extremal process. All point of the PPP are mapped to different points in the added dimension, so that they can easily be distinguished. All cluster points are mapped to the same point in the additional dimension as the PPP to which it belongs , see Figure 17.

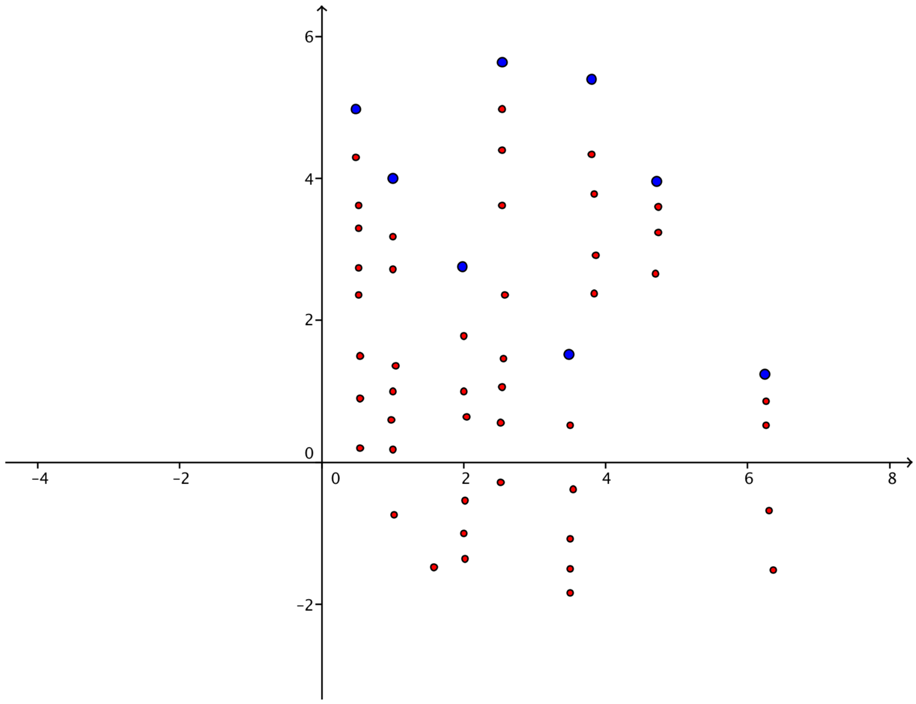


FIGURE 17. The extended extremal process of BBM. The red dots represent the Poisson points and the blue dots the cluster processes. Observe that all cluster points belonging to one Poisson point, have the same x-coordinate value as the corresponding Poisson point.

7. Complex temperature branching Brownian motion energy model

Let us continue by giving a brief summary of the content of Chapter 5, which is published as [52]. Before actually explaining our result we introduce the key elements that appear. Let $G_n = (V_n, E_n)$ be a sequence of graphs with vertex set V_n and edge set E_n. Suppose that G_n is finite for $n \in \mathbb{N}$. Moreover, let $H_{G_n} : V_n \to \mathbb{R}$ be a function, called *Hamiltonian*, measuring the energy of each configuration $\nu \in V_n$. A crucial goal in statistical mechanics is to define a probability measure (also in the limit as $n \uparrow \infty$) on the set of configurations, called *Gibbs measure*. For a finite Graph G_n the Gibbs measure $\mu_{\beta,G_n}$is given by

$$\mu_{\beta,G_n}(\nu) = \frac{\exp\left(-\beta H_{G_n}(\nu)\right)}{Z_{\beta,G_n}}, \quad \nu \in V_n, \beta \in \mathbb{R}. \tag{7.1}$$

β is called inverse temperature and Z_{β,G_n} is the normalization factor to turn μ_{β,G_n} into a probability measure. More precisely,

$$Z_{\beta,G_n} = \sum_{\nu \in V_n} \exp\left(-\beta H_{G_n}(\nu)\right) \tag{7.2}$$

and it is called *partition function*. Interestingly, Z_{β,G_n} contains a large amount of information. For example, it tells which configurations are likely to be observed at inverse temperature β. Generally speaking there are two different phenomena that can occur

(1) No single configuration has a positive probability to occur in the limit $n \uparrow \infty$.
(2) In the limit $n \uparrow \infty$, the measure μ_{β,G_n} concentrates a.s. on a finite set of configurations .

In the study of disordered systems one considers settings where the Hamiltonian itself is a random field. This makes the analysis much more involved and the two possibilities above turn into the following question

Does Z_{β,G_n} fulfil some kind of law of large numbers as n tends to infinity or is its behaviour dominated by the (random) extreme values of the Hamiltonian?

To get a glimpse how the above question should be answered, simplified models were introduced where the original Hamiltonian is replaced by a random field with a more tractable correlation structure. The simplest one is obtained by replacing the Hamiltonian by i.i.d. Gaussian random variables. On the hypercube $\Sigma_n = \{-1,1\}^n$ this leads to the random energy model (see also Section 2.2).

In [52] we aim at considering the model, where

$$V_t = \{1, \ldots, n(t)\}, \quad H_{V_t}(k) = x_k(t), \quad k \in V_t, \tag{7.3}$$

and $\{x_k(t), k \leq n(t)\}$ are the particle of a branching Brownian motion at time t. Moreover, we allow for complex temperatures $\beta \in \mathbb{C}$.

The BBM energy model is expected to be in the same universality class as Gaussian multiplicative chaos (see [67] for a review). There are several physical motivations to study models at complex temperature e.g. the construction of conformally invariant operators of 2d-string theory. Moreover, these models can be seen as a toy model for the Chalker-Coddington (CC) model which was introduced to understand the quantum Hall effect (see [67] for an overview). The results on the extremal process of BBM obtained in [1, 4]

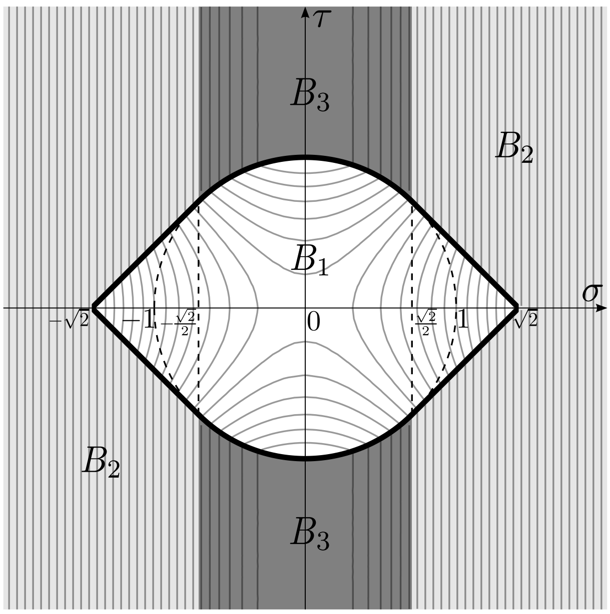


FIGURE 18. Expected phase diagram for the BBM energy model. The glassy phase is B_3 (Picture by A. Klimovsky).

turn out to be useful to answer this questions in the glassy phase. In the glassy phase (B_2 in Figure 18) the behaviour of the model is determined by the extreme values of the energy landscape. This means that in the limit $t \uparrow \infty$ the limiting partition function should only give weights to configurations with extremal energies. We would like to understand this rigorously. In [59] the convergence of the partition function in the glassy phase was proven when the imaginary and real parts of the energy are independent. In Chapter 5, respectively [52], we allow for arbitrary correlations and prove the convergence of the partition function:

> In the glassy phase B_2 the complex temperature partition function converges to a non-trivial limit when it is rescaled by $e^{\beta m(t)}$. $m(t)$ is as before the order a the maximum of a BBM, namely $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$. Moreover, the limiting random variables only emerges from extremal particles of BBM.

The precise statement can be found in Chapter 5, Theorem 1.1 and Theorem 1.2. The structure of the proof is as follows. In a first step we show that all particles below $m(t)-d$ for some constant d do not contribute to the limiting partition function as $d\uparrow\infty$. This is done by a refined second moment computation. To handle arbitrary correlation we use the following trick. If $X(t)$ and $Y(t)$ are two BBM's on the same Galton-Watson tree and for all $k=1,\dots,n(t)$

$$\mathrm{Cov}(x_k(t), y_k(t)) = |\rho| t, \tag{7.4}$$

then

$$y_k(t) \stackrel{d}{=} \rho x_k(t) + \sqrt{1-\rho^2} z_k(t), \tag{7.5}$$

where $Z(t) = (z_1(t),\dots,z_{n(t)})$ are the particles of a BBM on again the same Galton Watson tree but which is conditional on the Galton-Watson tree independent from $X(t)$ To prevent the moments from blowing up we use the localization of paths of extremal particles of BBM obtained in [3].

This implies that only extremal particles can contribute to the limiting partition function. We use a continuous mapping theorem together with the structure of the extremal process obtained in [1, 4] to determine the limiting partition function.

8. Beyond BBM

At the end of this introduction let me say a few words about related models. In fact, during the last few years there has been (and still is) a lot of interest in random fields whose correlation structure is similar to the one of BBM. All these models are expected to exhibit a certain common behaviour concerning extremal particles/points. The following paragraph should give a quick overview of a larger class of models where the phenomena described for BBM are expected to occur.

On the one hand there are log-correlated Gaussian free fields that have a similar correlation structure. E.g. in [25] Bramson et al. prove the convergence of the rescaled maximum in the two-dimensional Gaussian free field and In [36] further properties of the extremal points were established. In [13] Biskup and Louidor prove that the local maxima converge to a Poisson point process. More recently, they have been able to identify the complete extremal process [2]. On the other hand there are models e.g. related to cover times of Brownian motion on the torus. In [7] Belius and Kistler are able to identify the subleading order of the time to cover the torus with a Wiener sausage of size ϵ, generalizing results by Dembo et. al. [30] and Ding [35]. To identify this subleading order,

[2] Private communication by Marek Biskup and Oren Louidor. The article is still in preparation.

they identify a hidden branching structure and then implement first and second moment methods.

Another model is the randomized Riemann zeta function, which has been studied by Arguin et al. in [2]. They are also able to establish a hidden branching structure to get the leading order of the maximum. There are also conjectures on the characteristic polynomial of GUE random matrices (cf. [44]).

Hence, the techniques and ideas developed in the detailed study of (variable speed) branching Brownian motion should not be limited to this particular setting and should also help investigating further models.

9. Ageing at the critical temperature in the Random Energy Model

So far we have discussed questions related to equilibrium statistical mechanics. All limits we considered were concerned with taking the size of the system to infinity. In Section 5 we introduced the notion of Gibbs measure in (7.1). There we focused on understanding which type of configurations are likely to be observed. Now, we want to study how transitions between different configurations work and analyse the long time behaviour of the resulting process.

Abstractly speaking, the process is some Markov jump process in random environment. We will make this more precise after explaining the general motivation. The main question is

How does this Markov process behave for large times (and large space)?

In mathematical terms this question is often phrased the following way

What is the probability that the (rescaled) process is in the same position at some time t_0 and at a later time $t_0 + t_w$?

This probability is crucial to study the *ageing phenomenon*. A system is said to age if this probability depends on the initial waiting time t_0. The models where the ageing phenomenon is best understood are so-called *trap models*. They where introduced by Bouchaud and Dean in [15, 16].

They are Markov jump processes in random environment. We first introduce the key objects in a more general framework and then turn to the precise setting of Chapter 6. Let $G_n = (V_n, E_n)$ be a sequence if graphs with vertex set V_n and edge set E_n. We associate to each $x \in V_n$ a a positive random variable $\tau_n(x)$. The family of random variables $\{\tau_n(x), x \in V_n\}$ is defined on some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Next, let J_n be the Markov chain with initial distribution μ_n and invariant measure π_n, that jumps in each step uniformly at random to one of its neighbours. This choice of the jump chain J_n is referred to as Glauber dynamics. Recently there has been some progress in understanding Metropolis dynamics, where the jump rates depend heavily on the random environment [47, 27]. We denote by $p_n(x, y)$ for $x, y \in V_n$ the transition probabilities of J_n. The stochastic process $X_n(t)$ we can describe informally as follows. We start from some initial vertex $x \in V_n$. Then it waits there an exponential time with mean $\tau_n(x)$, before choosing one of its neighbours uniformly at random. To this vertex y the process jumps and waits there again an exponential time with mean $\tau_n(y)$ and so on.

To make this construction more precise, we define the *clock process* by

$$S_n(k) = \sum_{i=0}^{k} \tau_n\left(J_n(i)\right) e_{n,i}, \quad k \in \mathbb{N}, \tag{9.1}$$

where $\{e_{n,i}, i, n \in \mathbb{N}\}$ is a collection of independent exponentially distributed random variables with mean one. Moreover, they are also assumed to be independent from J_n and τ_n. Then

$$X_n(t) = J_n(i) \quad \text{if} \quad S_n(i) \le t \le S_n(i+1). \tag{9.2}$$

A time-time correlation function, $\mathcal{C}_n(t_0, t_w)$, $t_0, t_w \ge 0$: this is a function that quantifies the correlation between the state of the system at time t, $X_n(t_0)$, and its state at time t_0+t_w, $X_n(t_0+t_w)$.

A natural choice of correlation function, in view of ageing results in the REM, is

$$\mathcal{C}_n(t_0, t_0+t_w) = \mathcal{P}_{\mu_n}\left(X_n(t_0) = X_n(t_0+t_w)\right) \tag{9.3}$$

This probability can be rewritten using the clock process S_n.

$$\mathcal{C}_n(t_0, t_0+t_w) = \mathcal{P}\left(\{S_n(k), k \in \mathbb{N}\} \cap (t_0, t_0+t_w) = \emptyset\right), 0 \le t_0 < t_0+t_w\,. \tag{9.4}$$

To analyse this question a good understanding of the extrema of the underlying random energy landscape is necessary. It is likely that the Markov process spends a lot of time in these points and in particular if it reaches such an extremal point it will stay there for a long period of time. In Chapter 6 we study a particular model, namely the random energy model (REM). There, we make the following choice.

$$G_n = \Sigma_n = \{-1, 1\}^n \tag{9.5}$$

and

$$\tau_n(x) = e^{-\beta\sqrt{n}Z(x)}, \tag{9.6}$$

where $\{Z(x), x \in \Sigma_n\}$ are i.i.d. $\mathcal{N}(0,1)$-distributed. We start from the initial distribution, which is given by

$$\pi_n(x) = 2^{-n} \quad \forall x \in \Sigma_n. \tag{9.7}$$

Again, β is called inverse temperature and it is known that there is a critical temperature β_c at which the behaviour of the system changes dramatically. In the low temperature regime $\beta < \beta_c$ it has been proven that the system ages (see [8, 9]) and that is belongs to the so-called arcsine ageing regime (see [10]). In the analysis a deep understanding of random variables that are in the domain of attraction of an α-stable law with $\alpha \in (0;1)$ is necessary. In particular, this allows to choose the timescales on which ageing can be observed.

This connection was further emphasized by Gayrard in [46] and [45]. There the techniques by Durrett and Resnick [37], that ensure the convergence to α-stable subordinators, are used to establish the convergence of the rescaled clock process S_n.

The case where $\beta = \beta_c$ has been mainly left open. In [12] the behaviour of the correlation function was conjectured. It is not surprising that in this case the analysis of random variables that are in the domain of attraction of a 1-stable law is important. The simplest situation that is just concerned with a sum of i.i.d. random variables that are in the domain of attraction of a 1-stable law was treated in [38].

In Chapter 6 we prove the following when $\beta = \beta_c$ and corresponding timescales a_n, c_n (satisfying an additional technical condition).

> With the choice of correlation function as in (9.3), we have, for $t_0 = c_n t$ and $t_w = c_n s$ for $t, s > 0$, that $\sqrt{n}\mathcal{C}_n(t_0, t_0+t_w)$ converges either $\mathbb{P}$-a.s or in $\mathbb{P}$-probability to a non-trivial limit.

The precise statement can be found in Chapter 6, Theorem 1.3. This can intuitively be explained as follows. At the critical temperature the Markov chain spends in total more time (of order $\sqrt{n}$) in traps that have a low energy than in ones that are big. But to observe the event that the two-point correlation function describes we have to observe a big trap, where the process stays for a long time. This leads to a decay of the probability that is of order $1/\sqrt{n}$.

To conclude the introduction, let us build a bridge to the previous sections. It would be desirable to understand the ageing phenomena rigorously in models with highly correlated energy landscapes. Through the time change of J_n the dynamics are strongly influenced by the extreme values of the energy landscape. Hence, a precise understanding of the extremal process of the Hamiltonian can be seen as a first step in this direction. There is actually some hope to use the results (in particular of Chapter 2-4) to understand rigorously certain dynamics on these energy landscapes.

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CHAPTER 2

Two-speed Branching Brownian Motion

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The extremal process of two-speed branching Brownian motion*

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Abstract

We construct and describe the extremal process for variable speed branching Brownian motion, studied recently by Fang and Zeitouni [11], for the case of piecewise constant speeds; in fact for simplicity we concentrate on the case when the speed is σ_1 for $s \leq bt$ and σ_2 when $bt \leq s \leq t$. In the case $\sigma_1 > \sigma_2$, the process is the concatenation of two BBM extremal processes, as expected. In the case $\sigma_1 < \sigma_2$, a new family of cluster point processes arises, that are similar, but distinctively different from the BBM process. Our proofs follow the strategy of Arguin, Bovier, and Kistler in [3].

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1 Introduction

A standard branching Brownian motion (BBM) is a continuous-time Markov branching process that is constructed as follows: start with a single particle which performs a standard Brownian motion $x(t)$ with $x(0) = 0$ and continues for a standard exponentially distributed holding time T, independent of x. At time T, the particle splits independently of x and T into k offspring with probability p_k, where $\sum_{i=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$ and $K = \sum_{k=1}^{\infty} k(k-1)p_k < \infty$. These particles continue along independent Brownian paths starting from $x(T)$ and are subject to the same splitting rule. And so on.

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Branching Brownian motion has received a lot of attention over the last decades, with a strong focus on the properties of extremal particles. We mention the seminal contributions of McKean [18], Bramson, Lalley and Sellke, and Chauvin and Rouault [7, 6, 15, 8] on the connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation and on the distribution of the rescaled maximum. In recent years, their has been a revival of interest in BBM with numerous contributions, including the construction of the full extremal process [3, 1]. For a review of these developments see, e.g., the recent survey by Guéré [13].

BBM can be seen as a Gaussian process with covariances depending on an ultrametric distance, in this case the ultrametric associated to the genealogical structure of an underlying Galton-Watson process. In that respect it is closely related to another class of Gaussian processes, the Generalised Random Energy Models (GREM) introduced by Derrida and Gardner [12]. While in BBM the covariance of the process is a linear function of the ultrametric distance, in the GREM one considers more general functions. One of the reasons that makes BBM interesting in this context is the fact that the linear function appears as a borderline where the correlation starts to modify the behaviour of extremes [4, 5].

In the context of BBM, different covariances can be achieved by varying the speed (i.e. diffusivity) of the Brownian motions as a function of time (see also [5]). This model was introduced by Derrida and Spohn [9] and has recently been investigated by Fang and Zeitouni [11, 10], see also [16, 17]. The entire family of models obtained as time changes of BBM is a splendid test ground to further develop the theory of extremes of correlated random variables. Understanding fully the possible extremal processes that arise in this class should also provide us with candidate processes for even wider classes of random structures.

1.1 Results

In [11], Fang and Zeitouni showed that in the case when the covariance is a piecewise linear function, the maximum of BBM is tight and behaves as expected from the analogous GREM. In this paper we refine and extend their analysis: we obtain the precise law of the maximum, and we give the full characterisation of the extremal process.

For simplicity we consider the following variable speed BBM. Fix a time t. Then we consider the BBM model where at time s, all particles move independently as Brownian motions with variance

$$\sigma^2(s) = \begin{cases} \sigma_1^2 & 0 \le s < bt \\ \sigma_2^2 & t \le s \le t \end{cases}, \quad 0 < b \le 1. \tag{1.1}$$

We normalise the total variance by assuming

$$\sigma_1^2 b + \sigma_2^2 (1-b) = 1. \tag{1.2}$$

Note that in the case $b = 1$, $\sigma_2 = \infty$ is allowed.

We denote by $n(s)$ the number of particles at time s and by $\{x_i(s); 1 \le i \le n(s)\}$ the positions of the particles at time s.

Remark 1.1. *Strictly speaking, we are not talking about a single stochastic process, but about a family $\{x_k(s), k \le n(s)\}_{s \le t}^{t \in \mathbb{R}_+}$ of processes with finite time horizon, indexed by that horizon, t.*

In this case, Fang and Zeitouni [10] showed that

$$\max_{k \le n(t)} x_k(t) = \begin{cases} \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t + O(1), & \text{if } \sigma_1 < \sigma_2, \\ \sqrt{2}t(b\sigma_1 + (1-b)\sigma_2) - \frac{3}{2\sqrt{2}}(\sigma_1 + \sigma_2) \log t + O(1), & \text{if } \sigma_1 > \sigma_2. \end{cases} \tag{1.3}$$

The second case has a simple interpretation: the maximum is achieved by adding to the maxima of BBM at time bt the maxima of their offspring at time $(1-b)t$ later. The first case looks simpler even, but is far more interesting. The order of the maximum is that of the REM, a fact to be expected by the corresponding results in the GREM (see [12, 4]). But what is the law of the rescaled maximum and what is the corresponding extremal process? The purpose of this paper is primarily to answer this question.

For standard BBM, $\bar{x}(t)$, (i.e. $\sigma_1 = \sigma_2$), Bramson [7] and Lalley and Sellke [15] show that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} \bar{x}_k(t) - m(t) \leq y\right) = \omega(x) = \mathbb{E}e^{-CZe^{-\sqrt{2}y}}, \tag{1.4}$$

where $m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$, Z is a random variable, the limit of the so called *derivative martingale*, and C is a constant.

In [3] (see also [1] for a different proof) it was shown that the extremal process,

$$\lim_{t\uparrow\infty} \widetilde{\mathcal{E}}_t \equiv \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\bar{x}_k(t)-m(t)} = \widetilde{\mathcal{E}}, \tag{1.5}$$

exists in law, and $\widetilde{\mathcal{E}}$ is of the form

$$\widetilde{\mathcal{E}} = \sum_{k,j} \delta_{\eta_k + \Delta_j^{(k)}}, \tag{1.6}$$

where η_k is the k-th atom of a mixture of Poisson point process with intensity measure $CZe^{-\sqrt{2}y}dy$, with C and Z as before, and $\Delta_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t) - \max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{1.7}$$

where $\tilde{x}(t)$ is BBM conditioned on $\max_{j\leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}t$.

The main result of the present paper is similar but different.

Theorem 1.2. *Let $x_k(t)$ be branching Brownian motion with variable speed $\sigma^2(s)$ as given in (1.1). Assume that $\sigma_1 < \sigma_2$. Then*

(i)

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right) = \mathbb{E}e^{-C'Ye^{-\sqrt{2}y}}, \tag{1.8}$$

where $\tilde{m}(t) = \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t$, C' is a constant and Y is a random variable that is the limit of a martingale (but different from Z!).

(ii) The point process

$$\mathcal{E}_t \equiv \sum_{k\leq n(t)} \delta_{x_k(t)-\tilde{m}(t)} \to \mathcal{E}, \tag{1.9}$$

as $t\uparrow\infty$, in law, where

$$\mathcal{E} = \sum_{k,j} \delta_{\eta_k + \sigma_2\Lambda_j^{(k)}}, \tag{1.10}$$

where η_k is the k-th atom of a mixture of Poisson point process with intensity measure $C'Ye^{-\sqrt{2}y}dy$, with C' and Y as in (i), and $\Lambda_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Lambda^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t) - \max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{1.11}$$

where $\tilde{x}(t)$ is BBM of speed 1 conditioned on $\max_{j\leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}\sigma_2 t$.

To complete the picture, we give the result for the limiting extremal process in the case $\sigma_1 > \sigma_2$. This result is much simpler and totally unsurprising.

Theorem 1.3. *Let $x_k(t)$ be as in Theorem 1.1, but $\sigma_2 < \sigma_1$. Let $\widetilde{\mathcal{E}} \equiv \widetilde{\mathcal{E}}^0$ and $\widetilde{\mathcal{E}}^{(i)}, i \in \mathbb{N}$ be independent copies of the extremal process (1.6) of standard branching Brownian motion. Let*

$$m(t) \equiv \sqrt{2}t(b\sigma_1 + (1-b)\sigma_2) - \frac{3}{2\sqrt{2}}(\sigma_1+\sigma_2)\log t - \frac{3}{2\sqrt{2}}(\sigma_1 \log b + \sigma_2 \log(1-b)), \quad (1.12)$$

and set

$$\mathcal{E}_t \equiv \sum_{k \le n(t)} \delta_{x_k(t)-m(t)}. \quad (1.13)$$

Then

$$\lim_{t\uparrow\infty} \mathcal{E}_t = \mathcal{E}, \quad (1.14)$$

exists in law, and

$$\mathcal{E} = \sum_{i,j} \delta_{\sigma_1 e_i + \sigma_2 e_j^{(i)}}, \quad (1.15)$$

where $e_i, e_j^{(i)}$ are the atoms of the point processes $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}^{(i)}$, respectively.

Remark 1.4. *In the case $\sigma_1 < 1$, we see that the limiting process depends only on the values of σ_1 (through the martingale Y) and on σ_2 (through the processes of clusters $\sigma_2\Lambda^{(k)}$). As σ_2 grows, the clusters become spread out, and in the limit $\sigma_2 = \infty$, the cluster processes degenerate to the Dirac mass at zero. Hence, in that case the extremal process is just a mixture of Poisson point processes. When $\sigma_1 = 0$, and $b > 0$, the martingale limit is just an exponential random variable, the limit of the martingale $n(t)e^{-t}$. The case $b = 1$, $\sigma_1 = 0$ corresponds to the random REM, where there is just a random number of iid random variables of variance one present.*

Remark 1.5. *We have decided to write this paper only for the case of two speeds. It is fairly straightforward to extend our results to the general case of piecewise constant speed with a fixed number of change points. The details will be presented in a separate paper [14]. The general case of variable speed still offers more challenges, in spite of recent progress [16, 17].*

1.2 Outline of the proof

The proof of our result follows the strategy used in [3]. The main difference is that we show that particles that will reach the level of the extremes at time t must, at the time of the speed change, tb, lie in a $\sqrt{t}$-neighbourhood of a value $\sqrt{2}(\sigma_2-1)bt$ below the straight line of slope $\sqrt{2}$. This is the done in Section 2. Then two pieces of information are needed: in Section 3 we get precise bounds on the probabilities of BBM to reach values at excessively large heights, and more generally we control the behaviour of solutions of the F-KPP equations very much ahead of the travelling wave front. The final results comes from combining this information with the precise distribution of particles at the time of the speed change. This is done in Section 4 by proving the convergence of a certain martingale, analogous, but distinct from the derivative martingale that appears in normal BBM. The identification and the proof of L^1 convergence of this martingale is the key idea. Using this information in Sections 5 and 6, the convergence of the maximums, respectively the Laplace functional of the extremal process are proven, much along the lines on [3]. Section 7 provides various characterisations of the limiting process, as in [3]. In particular, we describe the extremal process in terms of an auxiliary process, constructed from a Poisson point process with a strange intensity to those atoms we add BBM's with negative drift. Interestingly, the process of the cluster extremes of this auxiliary process is again Poisson with random intensity driven by the new martingale. The results stated above follow then from looking at the clusters from their maximal points. In the final Section 8, we give the simple proof of Theorem 1.3

2 Localisation of paths

The key to understanding the behaviour of the two speed BBM is to control the positions of particle at time bt which are in the top at time t. This is done using Gaussian estimates.

Proposition 2.1. *Let $\sigma_1 < \sigma_2$. For any $d \in \mathbb{R}$ and any $\epsilon > 0$, there exists a constant $A > 0$ such that for all t large enough*

$$\mathbb{P}\left[\exists_{j\leq n(t)} \text{ s.t. } x_j(t) > \tilde{m}(t) - d \text{ and } x_j(bt) - \sqrt{2}\sigma_1^2 bt \notin [-A\sqrt{t}, A\sqrt{t}]\right] \leq \epsilon. \tag{2.1}$$

Proof. Using a first order Chebyshev inequality we bound (2.1) by

$$\begin{aligned} & e^t \mathbb{E}\left[\mathbb{1}_{\left\{\sigma_1\sqrt{bt}w_1 - \sqrt{2}\sigma_1^2 bt \notin [-A\sqrt{t}, A\sqrt{t}]\right\}} \mathbb{P}_{w_2}\left(\sigma_2\sqrt{(1-b)t}w_2 > \tilde{m}(t) - d - \sigma_1\sqrt{bt}w_1\right)\right] \\ = \; & e^t \mathbb{E}\Big[\mathbb{1}_{\{w_1 - \sqrt{2}\sigma_1\sqrt{bt} \notin [-A', A']\}} \mathbb{P}_{w_2}\Big(w_2 > \tfrac{\sqrt{2}t - \sigma_1\sqrt{b}w_1}{\sigma_2\sqrt{1-b}} - \tfrac{\log t}{2\sqrt{2}\sigma_2\sqrt{(1-b)t}} - \tfrac{d}{\sigma_2\sqrt{(1-b)t}}\Big)\Big] \\ \equiv \; & (R1) + (R2), \end{aligned} \tag{2.2}$$

where w_1, w_2 are independent $\mathcal{N}(0,1)$-distributed, $A' = \frac{1}{\sqrt{b}\sigma_1} A$, $\mathbb{P}_{w_2}$ denotes the law of the variable w_2. Introducing into the last line the identity in the form

$$1 = \mathbb{1}_{\{\sqrt{2}t - \sigma_1\sqrt{b}w_1 < \log t\}} + \mathbb{1}_{\{\sqrt{2}t - \sigma_1\sqrt{b}w_1 \geq \log t\}} \tag{2.3}$$

we can write it as $(R1) + (R2)$.

We first show $\lim_{t\to\infty}(R1) = 0$. Using the standard Gaussian tail estimate

$$\int_u^\infty e^{-x^2/2} dx \leq u^{-1} e^{-u^2/2}, \tag{2.4}$$

$(R1)$ is bounded from above by

$$e^t \mathbb{P}\left[\sqrt{2}t - \sigma_1\sqrt{b}w_1 < \log t\right] \leq e^{t(1-1/b\sigma_1^2) + t^{1/2}\log t/b\sigma_1^2} \to 0 \quad \text{as } t \to \infty. \tag{2.5}$$

For (R2) we can use again (2.4) to show that (R2) is smaller than

$$\begin{aligned} & e^t (2\pi)^{-1} \int_{\substack{w - \sqrt{2}\sigma_1\sqrt{bt} \notin [-A', +A'] \\ \sqrt{2}t - \sigma_1\sqrt{b}w_1 \geq \log t}} \frac{e^{-w^2/2}}{\frac{\sqrt{2}t}{\sqrt{1-b}\sigma_2} - \frac{\sigma_1\sqrt{b}}{\sigma_2\sqrt{1-b}} w} \\ & \times \exp\left(-\tfrac{1}{2}\left(\tfrac{\sqrt{2}t - \sigma_1\sqrt{b}w - \log t/(2\sqrt{2}\sqrt{t}) - d/\sqrt{t}}{\sqrt{1-b}\sigma_2}\right)^2\right) \mathrm{d}w. \end{aligned} \tag{2.6}$$

We change variables $w = \sqrt{2}\sigma_1\sqrt{bt} + z$. Then the integral in (2.6) can be bounded from above by

$$\frac{M}{\sqrt{2\pi\sigma_2^2(1-b)}} \int_{z \notin [-A', A']} e^{-\frac{z^2}{2\sigma_2^2(1-b)}} \mathrm{d}z, \tag{2.7}$$

where M is some positive constant. (2.7) can be made as small as desired by taking A (and thus A') sufficiently large. ◻

Remark 2.2. *The point here is that since $\sigma_1^2 < \sigma_1$, these particles are way below $\max_{k\leq n(bt)} x_k(bt)$, which is near $\sqrt{2}\sigma_1 bt$. The offspring of these particles that want to be top at time will have to race much faster (at speed $\sqrt{2}\sigma_2^2$, rather than just $\sqrt{2}\sigma_2$) than normal. Fortunately, there are lots of particles to choose from. We will have to control precisely how many.*

We need a slightly finer control on the path of the extremal particle until the time of speed change. To this end we define two sets on the space of paths, $X : \mathbb{R}_+ \to \mathbb{R}$, The first controls that the position of the path is in a certain tube up to time s and the second the position of the particle at time s.

$$\begin{aligned}\mathcal{T}_{s,r} &= \left\{X \middle| \forall_{0\leq q\leq s} |X(q) - \tfrac{q}{s}X(s)| \leq ((q\wedge(s-q))\vee r)^\gamma\right\}\\ \mathcal{G}_{s,A,\gamma} &= \left\{X \middle| X(s) - \sqrt{2}\sigma_1^2 s \in [-As^\gamma, +As^\gamma]\right\}\end{aligned} \tag{2.8}$$

Recall [7] that the ancestral path form 0 to $x_k(s)$ can be written as $x_k(q) = \frac{q}{s}x_k(s) + \mathfrak{z}_k(s)$, where $\mathfrak{z}_k$ is a Brownian bridge from 0 to 0 in time s, independent of $x_k(s)$. We need the following simple fact about Brownian bridges.

Lemma 2.3. *Let $\mathfrak{z}(q)$ be a Brownian bridge starting in zero and ending in zero at time s. Then for all $\gamma > 1/2$, the following is true. For all $\epsilon > 0$ there exists r such that*

$$\lim_{s\uparrow\infty} \mathbb{P}\left(|\mathfrak{z}(q)| < ((q\wedge(s-q))\vee r)^\gamma, \forall\, 0\leq q\leq s\right) > 1-\epsilon. \tag{2.9}$$

Proposition 2.4. *Let $\sigma_1 < \sigma_2$. For any $d \in \mathbb{R}, A > 0$, $\gamma > \frac{1}{2}$ and any $\epsilon > 0$, there exists constants $B > 0$ such that, for all t large enough,*

$$\mathbb{P}\left[\exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \wedge x_j \notin \mathcal{G}_{b\sqrt{t},B,\gamma}\right] \leq \epsilon. \tag{2.10}$$

Proof. For B and t sufficiently large the probability in (2.10) is bounded from above by

$$\mathbb{P}\left[\exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \wedge x_j \notin \mathcal{T}_{bt,r}\right] \tag{2.11}$$

Let w_1 and w_2 be independent $\mathcal{N}(0,1)$-distributed random variables and $\mathfrak{z}$ a Brownian bridge starting in zero and ending in zero at time bt. Using a first moment method as in the proof of Proposition 2.1 together with the independence of the Brownian bridge from its endpoint, one sees that (2.11) is bounded from above by

$$\begin{aligned}&e^t\mathbb{E}\left[\mathbb{1}_{\{\sigma_1\sqrt{bt}w_1 - \sqrt{2}\sigma_1^2 bt\in[-A\sqrt{t},A\sqrt{t}]\}}\mathbb{P}_{w_2}\left(\sigma_2\sqrt{(1-b)t}w_2 > \tilde{m}(t) - d - \sigma_1\sqrt{bt}w_1\right)\right]\\ &\times\mathbb{P}\left[\mathfrak{z}\notin\mathcal{T}_{bt,r}\right] < \epsilon,\end{aligned} \tag{2.12}$$

where the last bound follows from Lemma 2.3 (with ϵ replaced by ϵ/M) and the bound (2.7) obtained in the proof of Proposition 2.1. □

Proposition 2.5. *Let $\sigma_1 < \sigma_2$. For any $d \in \mathbb{R}, A, B > 0$, $\gamma > \frac{1}{2}$ and any $\epsilon > 0$, there exists a constant $r > 0$ such that for all t large enough*

$$\begin{aligned}\mathbb{P}\Big[&\exists_{j\leq n(t)} : x_j(t) > \tilde{m}(t) - d \wedge x_j \in \mathcal{G}_{bt,A,\frac{1}{2}} \cap \mathcal{G}_{b\sqrt{t},B,\gamma}\\ &\wedge x_j(b\sqrt{t}+\cdot) - x_j(b\sqrt{t}) \notin \mathcal{T}_{b(t-\sqrt{t}),r}\Big] \leq \epsilon.\end{aligned} \tag{2.13}$$

Proof. The proof of this proposition is essentially identical to the proof of Proposition 2.4. □

3 Asymptotic behaviour of BBM

Let $\tilde{x}(t)$ denote a standard BBM. We are interested in the asymptotic behavior of

$$\mathbb{P}\left[\max_{1\leq i\leq n(t)} \bar{x}_i(t) > x + \sqrt{2}t\right] \tag{3.1}$$

for $x = at + b\sqrt{t}$, $a \in \mathbb{R}_+, b \in \mathbb{R}$. Recall that $\mathbb{P}\left(\max_{k\leq n(t)} \bar{x}_k(t) > x\right)$ is the solution of the F-KPP equation

$$\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + (1-u(t,x)) - \sum_{k=1}^{\infty} p_k (1-u(t,x))^k. \tag{3.2}$$

with initial condition $u(0,x) = \mathbb{1}_{x<0}$. We are more generally interested in the behaviour of solutions for such large values of x. The following proposition is an extension of Lemma 4.5 in [3] for these values of x.

Proposition 3.1. *Let u be a solution to the F-KPP equation with initial data satisfying*
(i) $0 \leq u(0,x) \leq 1$*;*
(ii) for some $h>0$, $\limsup_{t\to\infty} \frac{1}{t}\log\int_t^{t(1+h)} u(0,y)dy \leq -\sqrt{2}$*;*
(iii) for some $v>0$, $M>0$, $N>0$*, it holds that* $\int_x^{x+N} u(0,y)dy > v$ *for all* $x \leq -M$*;*
(iv) moreover, $\int_0^\infty u(0,y) y e^{2y} dy < \infty$*.*
Then we have for $x = at + o(t)$

$$\lim_{t\to\infty} e^{\sqrt{2}x} e^{x^2/2t} t^{1/2} u(t, x+\sqrt{2}t) = C(a), \tag{3.3}$$

where $C(a)$ is a strictly positive constant. The convergence is uniform for a in compact intervals.

Define for $r>0$ the function $\Psi(r,t,x+\sqrt{2}t)$ by

$$\Psi(r,t,x+\sqrt{2}t) = \tag{3.4}$$
$$\frac{e^{-\sqrt{2}x}}{\sqrt{2\pi(t-r)}} \int_0^\infty u(r, y+\sqrt{2}r) e^{\sqrt{2}y} e^{-\frac{(y-x)^2}{2(t-r)}} \left[1 - e^{-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)}\right] \mathrm{d}y.$$

Lemma 3.2. *For $x = at + o(t)$ we have, under the assumptions of Proposition 3.1,*

$$\lim_{t\to\infty} e^{\sqrt{2}x} e^{x^2/2t} t^{1/2} \Psi(r,t,x+\sqrt{2}t) \tag{3.5}$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-a^2 r/2} u(r, y+\sqrt{2}r) e^{(\sqrt{2}+a)y} \left(1 - e^{-2ay}\right) dy \equiv C(r,a).$$

The convergence is uniform for a in a compact set.

Proof. Using (3.4) we have

$$\begin{aligned}
&\lim_{t\to\infty} e^{\sqrt{2}x} e^{x^2/2t} t^{1/2} \Psi(r,t,x+\sqrt{2}t) \\
= \quad &\lim_{t\to\infty} \frac{\sqrt{t}}{\sqrt{2\pi(t-r)}} e^{x^2/2t} \int_0^\infty u(r,y+\sqrt{2}r) e^{\sqrt{2}y} e^{-\frac{(y-x)^2}{2(t-r)}} \\
&\times \left[1 - \exp\left(-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)\right)\right] \mathrm{d}y.
\end{aligned} \tag{3.6}$$

Next we show that we can use dominated convergence to take the limit $t \to \infty$ into the integral. First, the integrand is bounded by

$$Be^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a+1)y}, \tag{3.7}$$

where $B > 0$. As was shown by Bramson [6] (and used in [3]), the solution of the F-KPP equation can be bounded by the solution $u^{(2)}(t, x)$ of the linearised F-KPP equation

$$\partial_t u^{(2)} = \frac{1}{2}u^{(2)}_{xx} - u^{(2)} \tag{3.8}$$

with the same initial condition $u^{(2)}(0, x) = u(0, x)$. Moreover there exists y_0 such that for any $x > 0$

$$u^{(2)}(t, x) \le e^t e^{-x^2/2t} e^{y_0 x/t} \tag{3.9}$$

Thus we get that

$$\begin{aligned} &\int_0^\infty Be^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a+1)y}\mathrm{d}y \\ \le\ &\int_0^\infty B(r)e^{-a^2r/2}e^{-y^2/2r}e^{(a+1)y}\mathrm{d}y < \infty. \end{aligned} \tag{3.10}$$

where $B(r)$ is a constant that only depends on r. Hence we can apply dominated convergence to (3.6) and obtain

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_0^\infty u(r, y + \sqrt{2}r)e^{\sqrt{2}y}\lim_{t\to\infty}\left[e^{\sqrt{2}y}e^{-\frac{(y-x)^2}{2(t-r)}}\left[1 - e^{-2y\left(\frac{x+\frac{3}{2\sqrt{2}}\log t}{t-r}\right)}\right]\right]\mathrm{d}y \\ = &\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-a^2r/2}u(r, y + \sqrt{2}r)e^{(\sqrt{2}+a)y}\left(1 - e^{-2ay}\right)\mathrm{d}y. \end{aligned} \tag{3.11}$$

This proves the lemma. □

Proof of Proposition 3.1. Due to the assumptions (i),(ii),(iii) and (iv) we can use Proposition 4.3 of [3] for $t > 8r$ and $x > 8r - \frac{3}{2\sqrt{2}}\log t$ and r large enough:

$$\gamma^{-1}(r)\Psi(r, t, x + \sqrt{2}t) \le u(t, x + \sqrt{2}t) \le \gamma(r)\Psi(r, t, x + \sqrt{2}t), \tag{3.12}$$

where $\gamma(r)$ does not depend x and t and $\lim_{r\to\infty}\gamma(r) = 1$. Since $\gamma(r) \to 1$ as $r \to \infty$ this implies

$$\limsup_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \le \liminf_{r\to\infty} C(r, a) \tag{3.13}$$

and

$$\liminf_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \ge \limsup_{r\to\infty} C(r, a) \tag{3.14}$$

Hence $\lim_{r\to\infty} C(r, a) = C(a)$ exists. Moreover,

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \tag{3.15}$$

exists and is equal to $C(a)$. It is left to show that $C(a) \neq 0$ for $a > 0$. If $C(a) = 0$ we would have

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) = 0, \tag{3.16}$$

but

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \ge C(r, a)\gamma(r)^{-1}, \tag{3.17}$$

for r large enough, by (3.12). This contradicts (3.16). The same proposition implies

$$\lim_{t\to\infty} e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t, x + \sqrt{2}t) \le C(r, a)\gamma(r). \tag{3.18}$$

Hence $C(a) \neq \infty$. Proposition 3.1 is proven. □

4 The McKean martingale

In this section we pick up the idea of [15] and consider a suitable convergent martingale for the time inhomogeneous BBM with $\sigma_1 < \sigma_2$. Let $x_i(s)$, $1 \leq i \leq n(s)$ be the particles of a BBM where the Brownian motions have variance σ_1^2 with $\sigma_1^2 < 1$. Define

$$Y_s = \sum_{i=1}^{n(s)} e^{-s(1+\sigma_1^2)+\sqrt{2}x_i(s)}. \tag{4.1}$$

This turns out to be a uniformly integrable martingale that converges almost surely to a positive limit Y.

Remark 4.1. *Note that in terms of statistical mechanics, Y_s can be thought of as a normalised partition function at inverse temperature $\sigma_1\sqrt{2}$ (for ordinary BBM). Here the critical temperature is $\sqrt{2}$, so that we are in the high-temperature case. In the case of the GREM, where the underlying tree is deterministic, this quantity is known to even converge to a constant [4].*

Theorem 4.2. *The limit $\lim_{s\to\infty} Y_s$ exists almost surely and in L^1, is finite and strictly positive.*

The assertion of this theorem follows immediately from the following proposition.

Proposition 4.3. *If $\sigma_1 < 1$, Y_s is a uniformly integrable martingale with $\mathbb{E}[Y_s] = 1$*

Remark 4.4. *We would like to call this martingale* McKean martingale*, since McKean [18] had originally conjectured that this martingale (with $\sigma_1 = 1$) was the martingale in the representation of the extremal distribution of BBM, which, as Lalley and Sellke showed is wrong as it is actually the derivative martingale that appears there. We find it nice to see that in the time-inhomogeneous case with $\sigma_1 < 1$, KcKean turns out to be right! We will see in the proof that the uniform integrability of this martingale breaks down at $\sigma_1 = 1$.*

Remark 4.5. *Note further that if $\sigma_1 = 0$, then $Y_t = e^{-t}n(t)$ which is well known to converge to an exponential random variable.*

Proof. Clearly,

$$\mathbb{E}[Y_s] = e^s \mathbb{E}\left[e^{-(1+\sigma_1^2)s+\sqrt{2}x_1(s)}\right] = 1. \tag{4.2}$$

Next we show that Y_s is a martingale. Let $0 < r < s$. Then

$$\mathbb{E}[Y_s|\mathcal{F}_r] = \sum_{i=1}^{n(r)} \mathbb{E}\left[\sum_{j=1}^{n_j(s-r)} e^{-s(1+\sigma_1^2)+\sqrt{2}(x_j^i(s-r)+x_i(r))} \mid \mathcal{F}_r\right], \tag{4.3}$$

where for $1 \leq i \leq r$: $\left\{x_j^i(s-r), 1 \leq j \leq n_i(s-r)\right\}$ are the particles of independent BBM's with variance σ_1^2 at time $s-r$. (4.3) is equal to

$$\sum_{i=1}^{n(r)} e^{-r(1+\sigma_1^2)+\sqrt{2}x_i(r)} = Y_r, \tag{4.4}$$

as desired.

It remains to show that Y_s is uniformly integrable. We will write abusively $x_k(r)$ for the ancestor of $x_k(s)$ at time $r \leq s$ and write x_k for the entire ancestral path of $x_k(s)$. Define the truncated variable

$$Y_s^A = \sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2}, x_i \in \mathcal{T}_{s,r}\}}. \tag{4.5}$$

First $Y_s - Y_s^A \geq 0$, and a simple computation using the independence of $x_k(s)$ and $x_k(r) - \frac{r}{s}x_k(s)$ together with Lemma 2.3 shows that

$$\begin{aligned}\mathbb{E}\left[Y_s - Y_s^A\right] &\leq e^s \int_{-\infty}^{\infty} e^{-(1+\sigma_1^2)s+\sqrt{2s}\sigma_1 x} \mathbb{1}_{\{x-\sqrt{2s}\sigma_1 \notin [-A,A]\}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} + \epsilon \\ &= \int_{|z|>A} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} + \epsilon, \qquad (4.6)\end{aligned}$$

which can be made as small as desired by taking A and r to infinity. The key point is that the the second moment of Y_s^A is uniformly bounded in s.

$$\mathbb{E}\left[(Y_s^A)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right)^2\right] \equiv (T1) + (T2), \qquad (4.7)$$

where

$$\begin{aligned}(T1) &= \mathbb{E}\left[\sum_{i=1}^{n(s)} e^{-2((1+\sigma_1^2)s-\sqrt{2}x_i(s))} \mathbb{1}_{\{x_i \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right] \\ (T2) &= \mathbb{E}\left[\sum_{\substack{i,j=1 \\ i\neq j}} e^{-2(1+\sigma_1^2)s+\sqrt{2}(x_i(s)+x_j(s))} \mathbb{1}_{\{x_i,x_j \in \mathcal{G}_{s,A,1/2} \cap \mathcal{T}_{s,r}\}}\right] \qquad (4.8)\end{aligned}$$

We start by controlling $(T1)$.

$$\begin{aligned}(T1) &\leq \frac{e^{(s-2s(1+\sigma_1^2)}}{\sqrt{2\pi}} \int_{\sqrt{2s}\sigma_1 - A/\sigma_1}^{\sqrt{2s}\sigma_1 + A/\sigma_1} e^{2\sqrt{2s}\sigma_1 x} e^{-x^2/2} dx \\ &= \frac{e^{-(1-\sigma_1^2)s}}{\sqrt{2\pi}} \int_{-A/\sigma_1}^{A/\sigma_1} e^{-x^2/2} dx \leq e^{-(1-\sigma_1^2)s} \to 0 \quad \text{as } s \to \infty. \qquad (4.9)\end{aligned}$$

Now we control $(T2)$. By the sometimes so-called "many-to-two lemma" (see e.g.[6], Lemma 10), and dropping the useless parts of the conditions on the Brownian bridges

$$\begin{aligned}(T2) \leq \quad & Ke^s \int_0^s e^{s-q} \int_{\sqrt{2}\sigma_1^2 q - I_1(q,s)}^{\sqrt{2}\sigma_1^2 q + I_1(q,s)} \left(\int_{\sqrt{2}\sigma_1^2 s - A\sqrt{s} - x}^{\sqrt{2}\sigma_1^2 s + A\sqrt{s} - x} e^{-s(1+\sigma_1^2)+\sqrt{2}(x+y)} \right. \\ & \left. \times e^{-\frac{y^2}{2\sigma_1^2(s-q)}} \frac{dy}{\sigma_1\sqrt{2\pi(s-q)}} \right)^2 e^{-\frac{x^2}{2q\sigma_1^2}} \frac{dxdq}{\sqrt{2\pi\sigma_1^2 q}}, \qquad (4.10)\end{aligned}$$

where $K = \sum_{i=1}^{\infty} p_k k(k-1)$ and $I_1(q,s) = Aq/\sqrt{s} + ((q \wedge (s-q)) \vee r)^\gamma$. Moreover We change variables $x = z + \sqrt{2}\sigma_1^2 q$ and obtain

$$\begin{aligned}& Ke^s \int_0^s e^{s-q} \int_{-I_1(q,s)}^{+I_1(q,s)} \left(\int_{\sqrt{2}\sigma_1^2(s-q) - A\sqrt{s} - z}^{\sqrt{2}\sigma_1^2(s-q) + A\sqrt{s} - z} e^{-s(1+\sigma_1^2)+\sqrt{2}(z+\sqrt{2}\sigma_1^2 q + y)} \right. \\ & \left. \times e^{-\frac{y^2}{2\sigma_1^2(s-q)}} \frac{dy}{\sigma_1\sqrt{2\pi(s-q)}} \right)^2 e^{-\frac{(z+\sqrt{2}\sigma_1^2 q)^2}{2\sigma_1^2 q}} \frac{dzdq}{\sqrt{2\pi\sigma_1^2 q}}, \qquad (4.11)\end{aligned}$$

Now we change variables $w = \frac{y}{\sigma_1\sqrt{s-q}} - \sqrt{2}\sigma_1\sqrt{s-q}$. (4.11) is equal to

$$K \int_0^s e^{-q(1-2\sigma_1^2)} \int_{-I_1(q,s)}^{+I_1(q,s)} e^{+2\sqrt{2}z} \left(\int_{\frac{-A\sqrt{s}-z}{\sigma_1\sqrt{s-q}}}^{\frac{+A\sqrt{s}-z}{\sigma_1\sqrt{s-q}}} e^{-w^2/2} \frac{dw}{\sqrt{2\pi}} \right)^2 e^{-\frac{(z+\sqrt{2}\sigma_1^2 q)^2}{2\sigma_1^2 q}} \frac{dzdq}{\sqrt{2\pi\sigma_1^2 q}}. \qquad (4.12)$$

Now the integral with respect to w is bounded by 1. Hence (4.12) is bounded from above by

$$K\int_0^s e^{-q(1-2\sigma_1^2)}\int_{-I_1(q,s)/\sigma_1\sqrt{q}}^{+I_1(q,s)/\sigma_1\sqrt{q}} e^{-\frac{(z-\sqrt{2}\sigma_1\sqrt{q})^2}{2}}\frac{dzdq}{\sqrt{2\pi}}. \tag{4.13}$$

We split the integral over q into the three parts R_1, R_2, and R_3 according to the integration from 0 to r, r to $s-r$, and $s-r$ to r, respectively. Then

$$R_2 \le K\int_r^{s-r} e^{-q(1-2\sigma_1^2)}\frac{e^{-\frac{1}{2}\left(I_1(q,s)/\sigma_1\sqrt{q}-\sqrt{2}\sigma_1\sqrt{q}\right)^2}}{\sqrt{2\pi}\left(\sqrt{2}\sigma_1\sqrt{q}-I_1(q,s)/\sigma_1\sqrt{q}\right)}dq \tag{4.14}$$

This is bounded by

$$K\int_r^{s-r} e^{-(1-\sigma_1^2)q+O(q^\gamma)}dq \le \frac{C}{1-\sigma_1^2}e^{-c(1-\sigma_1^2)r}. \tag{4.15}$$

For R_1 the integral over z can only be bounded by one. This gives

$$R_1 \le K\int_0^r e^{(2\sigma_1^2-1)q}dq \equiv D_1(r), \tag{4.16}$$

R_3 can be treated the same way as R_2 and we get

$$R_3 \le K\int_{s-r}^s e^{-(1-\sigma_1^2)q+O(r^\gamma)}dq \le \frac{K}{1-\sigma_1^2}e^{-(1-\sigma_1^2)(s-r)+O(r^\gamma)} \to 0 \quad \text{as } s\to\infty. \tag{4.17}$$

Putting all three estimates together, we see that $\sup_s \mathbb{E}\left[\left(Y_s^A\right)^2\right] \le D_2(r)$. From this it follows that Y_s is uniformly integrable. Namely,

$$\begin{aligned}\mathbb{E}[Y_s\mathbb{1}_{Y_s>z}] &= \mathbb{E}[Y_s^A\mathbb{1}_{Y_s>z}]+\mathbb{E}[(Y_s-Y_s^A)\mathbb{1}_{Y_s>z}] \\ &= \mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s^A>z/2}\right]+\mathbb{E}\left[Y_s^A\left(\mathbb{1}_{Y_s>z}-\mathbb{1}_{Y_s^A>z/2}\right)\right]+\mathbb{E}[(Y_s-Y_s^A)\mathbb{1}_{Y_s>z}].\end{aligned} \tag{4.18}$$

For the first term we have

$$\mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s^A>z/2}\right] \le \frac{2}{z}\mathbb{E}\left[\left(Y_s^A\right)^2\right] \le \frac{2}{z}D_2(r). \tag{4.19}$$

For the second, we have

$$\begin{aligned}\mathbb{E}\left[Y_s^A\left(\mathbb{1}_{Y_s>z}-\mathbb{1}_{Y_s^A>z/2}\right)\right] &\le \mathbb{E}\left[Y_s^A\mathbb{1}_{Y_s-Y_s^A\ge z/2}\mathbb{1}_{Y_s^A\le z/2}\right] \\ &\le \frac{z}{2}\mathbb{P}\left[(Y_s-Y_s^A)>z/2\right] \le \mathbb{E}\left[Y_s-Y_s^A\right].\end{aligned} \tag{4.20}$$

The last term in (4.18) is also bounded by $\mathbb{E}\left[Y_s-Y_s^A\right]$. Choosing now A and r such that $\mathbb{E}\left[Y_s-Y_s^A\right]\le\epsilon/3$, and then z so large that $\frac{2}{z}D_2(r)\le\epsilon/3$, we obtain that $\mathbb{E}[Y_s\mathbb{1}_{Y_s>z}]\le\epsilon$, for large enough z, uniformly in s. Thus Y_s is uniformly integrable, which we wanted to show. □

Proof of Theorem 4.2. By Proposition 4.3 Y_s is a positive uniformly integrable martingale. By Doob's martingale convergence theorem we have that $\lim Y_s = Y$ exists almost surely and is finite. Moreover Y is positive and $Y_s \xrightarrow{L^1} Y$. In particular, this implies $Y \not\equiv 0$. □

We will also need to control the processes $\tilde{Y}_{s,\gamma}^A = \sum_{i=1}^{n(s)} e^{-(1+\sigma_1^2)s+\sqrt{2}x_i(s)}\mathbb{1}_{x_i\in\mathcal{G}_{s,A,\gamma}}$.

Lemma 4.6. *The family of random variables* $\tilde{Y}_{s,\gamma}^A$, $s, A \in \mathbb{R}_+, 1>\gamma>1/2$ *is uniformly integrable and converges, as* $s\uparrow\infty$ *and* $A\uparrow\infty$, *to* Y, *both in probability and in* L^1.

Proof. The proof of uniform integrability is a rerun of the proof of Proposition 4.3, noting that the bounds on the truncated second moments are uniform in A. Moreover, the same computation as in Eq. (4.6) shows that $\mathbb{E}|Y_s - \tilde{Y}^A_{s,\gamma}| \le \epsilon$, uniformly in s, for A large enough. Therefore,

$$\lim_{A\uparrow\infty} \limsup_{s\uparrow\infty} \mathbb{E}|Y_s - \tilde{Y}^A_{s,\gamma}| = 0, \tag{4.21}$$

which implies that $Y_s - \tilde{Y}^A_{s,\gamma}$ converges to zero in probability. Since Y_s converges to Y almost surely, we arrive at the second assertion of the lemma. □

5 Convergence of the maximum of two-speed BBM

Using the results established in the last three sections, we show now the convergence of the law the of the maximum of two-speed BBM in the case $\sigma_1 < \sigma_2$.

Theorem 5.1. *Let $\{x_k(t), 1 \le k \le n(t)\}$ be the particles of a time inhomogeneous BBM with $\sigma_1 < \sigma_2$ and the normalising assumption $\sigma_1^2 b + \sigma_2^2(1-b) = 1$. Then, with $\tilde{m}(t)$ as in Theorem 1.1,*

$$\lim_{t\to\infty} \mathbb{P}\left[\max_{1\le k\le n(t)} x_k(t) - \tilde{m}(t) \le y\right] = \mathbb{E}\left[\exp\left(-\sigma_2 C(a) Y e^{-\sqrt{2}y}\right)\right]. \tag{5.1}$$

Y is the limit of the McKean martingale from the last section, and $C(a)$ is the positive constant given by

$$C(a) = \lim_{r\to\infty} \int_0^\infty e^{-a^2 r/2} \mathbb{P}\left[\max_{k\le n(t)} \bar{x}_k(r) > z + \sqrt{2}r\right] e^{(\sqrt{2}+a)z}\left(1 - e^{-2az}\right) dz, \tag{5.2}$$

where $\{\bar{x}_k(t), k \le n(t)\}$ are the particles of a standard BBM and $a = \sqrt{2}(\sigma_2 - 1)$.

Proof. Denote by $\{x_i(bt), 1 \le i \le n(bt)\}$ the particles of a BBM with variance σ_1 at time bt and by $\mathcal{F}_{bt}$ the σ-algebra generated this BBM. Moreover, for $1 \le i \le n(bt)$, let $\{x^i_j((1-b)t), 1 \le j \le n_i((1-b)t)\}$ denote the particles of independent BBM with variance σ_2 at time $(1-b)t$.

Let us first observe that by the analog of Theorem 1.1. of [10] for two-speed BBM[1] we know that the maximum of our process is not too small, namely that for any $\epsilon > 0$, there exists $d < \infty$, such that

$$\mathbb{P}\left[\max_{1\le k\le n(t)} x_k(t) - \tilde{m}(t) \le -d\right] \le \epsilon/2. \tag{5.3}$$

Therefore,

$$\begin{aligned} \mathbb{P}\left[-d \le \max_{1\le k\le n(t)} x_k(t) - \tilde{m}(t) \le y\right] &\le \mathbb{P}\left[\max_{1\le k\le n(t)} x_k(t) - \tilde{m}(t) \le y\right] \\ &\le \mathbb{P}\left[-d \le \max_{1\le k\le n(t)} x_k(t) - \tilde{m}(t) \le y\right] \\ &+ +\epsilon/2 \end{aligned} \tag{5.4}$$

[1] As pointed out in [11], the arguments used for branching random walks carry all over to BBM.

On the other hand, by Proposition 2.1, we have that there exists $A < \infty$, such that

$$\begin{aligned} &\mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \in \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] \\ &\leq \mathbb{P}\left[-d \leq \max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] \\ &= \mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \in \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] \\ &+\mathbb{P}\left[\exists_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \notin \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] \\ &\leq \mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \in \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] + \epsilon/2 \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we have that

$$\begin{aligned} &\mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \in \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] \\ &\leq \mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\}\right] \\ &\leq \mathbb{P}\left[\forall_{1\leq k\leq n(t)}\{-d \leq x_k(t) - \tilde{m}(t) \leq y\} \cap \{x_k \in \mathcal{G}_{bt,A,\frac{1}{2}}\}\right] + \epsilon \end{aligned} \tag{5.6}$$

Thus we obtain

$$\begin{aligned} &\mathbb{P}\left[\max_{1\leq k\leq n(t)} x_k(t) - \tilde{m}(t) \leq y\right] \\ &= \mathbb{P}\left[\max_{1\leq i\leq n(bt)} \max_{1\leq j\leq n_i((1-b)t)} x_i(bt) + x_j^i((1-b)t) - \tilde{m}(t) \leq y\right] \\ &= \mathbb{E}\left[\prod_{1\leq i\leq n_i(bt)} \mathbb{P}\left[\max_{1\leq j\leq n_i((1-b)t)} x_j^i((1-b)t) \leq \tilde{m}(t) - x_i(bt) + y \mid \mathcal{F}_{bt}\right]\right] \\ &\leq \mathbb{E}\left[\prod_{\substack{1\leq i\leq n(bt)\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}}}} \mathbb{P}\left[\max_{1\leq j\leq n_i((1-b)t)} \sigma_2^{-1} x_j^i((1-b)t) \leq \sigma_2^{-1}\left(\tilde{m}(t) - x_i(tb) + y\right) \mid \mathcal{F}_{tb}\right]\right] \\ &+\epsilon. \end{aligned} \tag{5.7}$$

Of course the corresponding lower bound holds without the ϵ.

Observe that the last probability in (5.7) is equal to

$$1 - \mathbb{P}\left[\max_{1\leq j\leq n_i((1-b)t)} \bar{x}_j^i((1-b)t) > \sigma_2^{-1}\left(\tilde{m}(t) - x_i(tb) + y\right) \mid \mathcal{F}_{tb}\right], \tag{5.8}$$

where $\bar{x}_j^i((1-b)t)$ are the particles of a standard BBM. Using Proposition 3.1 for $(1-b)t$ and $u(t,x) = \mathbb{P}\left(\max \bar{x}_j^i(t) > x\right)$, and setting

$$C_t(x) \equiv e^{\sqrt{2}x + x^2/2t} t^{1/2} u(t, x + \sqrt{2}t), \tag{5.9}$$

we can write the probabilities in the last line of (5.8) as

$$\begin{aligned} &u\left((1-b)t, \sigma_2^{-1}(\tilde{m}(t) - x_i(bt) + y)\right) \\ &= C_{(1-b)t}\left(\sigma_2^{-1}(\tilde{m}(t) - x_i(bt) + y) - t\sqrt{2}(1-b)\right)\right) \\ &\times e^{-\sqrt{2}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2} - \sqrt{2}(1-b)t\right)} e^{-\frac{1}{2(1-b)t}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2} - \sqrt{2}(1-b)t\right)^2} ((1-b)t)^{-1/2} \end{aligned} \tag{5.10}$$

Now all the $x_i(bt)$ that appear are of the form $x_i(bt) = \sqrt{2}\sigma_1^2 bt + O(\sqrt{t})$, so that

$$C_{(1-b)t}\left(\sigma_2^{-1}(\tilde{m}(t) - x_i(bt) + y) - \sqrt{2}(1-b)t\right) = C_{(1-b)t}(a(1-b)t + O(\sqrt{t})), \tag{5.11}$$

with (using (1.2))

$$a \equiv \frac{1}{1-b}\left(\frac{\sqrt{2}-\sqrt{2}\sigma_1^2 b}{\sigma_2} - \sqrt{2}(1-b)\right) = \sqrt{2}(\sigma_2 - 1), \tag{5.12}$$

But then, by Proposition 3.1,

$$\lim_{t\uparrow\infty} C_{(1-b)t}\left(\sigma_2^{-1}(\tilde{m}(t) - x_i(bt) + y) - \sqrt{2}(1-b)t\right) = C(a), \tag{5.13}$$

with uniform convergence for all i appearing in (5.7) and $C(a)$ is the constant given by (5.2). Thus we can rewrite the expectation in (5.7) as

$$\begin{aligned}
&\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \mathbb{P}\left[\max_{1\le j\le n_i((1-b)t)} \sigma_2^{-1} x_j^i((1-b)t) \le \sigma_2^{-1}\left(\tilde{m}(t) - x_i(tb) + y\right) \mid \mathcal{F}_{tb}\right]\Bigg] \\
&= \mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \Big\{1 - C(a) e^{-\sqrt{2}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2} - \sqrt{2}(1-b)t\right)} \\
&\qquad \times e^{-\frac{1}{2(1-b)t}\left(\frac{\tilde{m}(t)-x_i(bt)+y}{\sigma_2} - \sqrt{2}(1-b)t\right)^2} ((1-b)t)^{-1/2}(1+o(1))\Big\}\Bigg].
\end{aligned} \tag{5.14}$$

This is equal to

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \Big\{1 - C(a)((1-b)t)^{-1/2} e^{(1-b)t - \frac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}} (1+o(1))\Big\}\Bigg]. \tag{5.15}$$

Using that $x_i(bt) - \sqrt{2}\sigma_1^2 tb \in [-A\sqrt{t}, A\sqrt{t}]$ we have the uniform bounds

$$\exp\left((1-b)t - \tfrac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}\right) \le \exp\left((1-\sigma_2^2)(1-b)t + \log t + A\sqrt{t}\right). \tag{5.16}$$

Observe that the right-hand side of Eq. (5.16)$\to 0$ as $t\uparrow\infty$, since $\sigma_2^2 > 1$. Hence (5.15) is equal to

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(bt)\\ x_i\in\mathcal{G}_{bt,A,1/2}}} \exp\left(-C(a)((1-b)t)^{-1/2} e^{(1-b)t - \frac{(\tilde{m}(t)+y-x_i(bt))^2}{2(1-b)t\sigma_2^2}} (1+o(1))\right)\Bigg]. \tag{5.17}$$

Expanding the square in the exponent in the last line and keeping only the relevant terms yields

$$\sqrt{2}y + t\sigma_2^2(1-b) + 2\sigma_1^2 bt - \sqrt{2}x_i(bt) + \frac{\left(\sqrt{2}t\sigma_1^2 b - x_i(bt)\right)^2}{2(1-b)\sigma_2^2 t}. \tag{5.18}$$

The terms up to the last one would nicely combine to produce the McKean martingale as coefficient of $C(a)$. However, the last terms are of order one and cannot be neglected. To deal with them, we split the process at time $b\sqrt{t}$. We write somewhat abusively $x_i(bt) = x_i(b\sqrt{t}) + x_l^{(i)}(b(t-\sqrt{t}))$, where we understand that $x_i(b\sqrt{t})$ is the ancestor at time $b\sqrt{t}$ of the particle that at time t is labeled i if we think backwards from time t, while the labels of the particles at time $b\sqrt{t}$ run only over the different ones, i.e. up to $n(b\sqrt{t})$, if we think in the forward direction. No confusion should occur if this is kept in mind.

Using Proposition 2.4 and Proposition 2.5 we can further localise the path of the particle. Recall the definition of $\mathcal{G}_{s,A,\gamma}$ and $\mathcal{T}_{r,s}$, we rewrite (5.17), up to a term of order ϵ, as

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(b\sqrt{t})\\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}}\mathbb{E}\Bigg[\prod_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{b(t-\sqrt{t}),r}}}\exp\Bigg(-C(a)((1-b)t)^{-1/2} \tag{5.19}$$
$$\times\exp\left((1-b)t-\tfrac{(\tilde{m}(t)+y-x_i(b\sqrt{t})-x_l^{(i)}(b(t-\sqrt{t})))^2}{2(1-b)t\sigma_2^2}\right)(1+o(1))\Bigg)\Big|\mathcal{F}_{\sqrt{t}b}\Bigg]\Bigg].$$

Using that $x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t}))-\sqrt{2}\sigma_1^2tb\in[-A\sqrt{t},A\sqrt{t}]$ and $\tilde{m}=\sqrt{2}-\frac{1}{2\sqrt{2}}\log t$, we can re-write the terms multiplying $C(a)$ in (5.19) as

$$\begin{aligned}&\exp\Big(-(1+\sigma_1^2)bt+\sqrt{2}(x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t})))-\frac{1}{2}\log(1-b)-\sqrt{2}y\\&-\tfrac{(x_i(b\sqrt{t})+x_l^{(i)}(b(t-\sqrt{t}))-\sqrt{2}\sigma_1^2bt)^2}{2(1-b)\sigma_2^2t}+O(1/\sqrt{t})\Big)\\&\equiv E(x_i,x_l^{(i)})=E(x_i(b\sqrt{t}),x_l^{(i)}(b(t-\sqrt{t})))=E(x_i(b\sqrt{t}),x_i(bt)-x_i(b\sqrt{t})).\end{aligned} \tag{5.20}$$

Now (5.19) takes the form

$$\mathbb{E}\Bigg[\prod_{\substack{1\le i\le n(b\sqrt{t})\\ x_i\in\mathcal{G}_{b\sqrt{t},B,\gamma}}}\mathbb{E}\Bigg[\exp\Big\{-\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})(1+o(1))\Big\}\Big|\mathcal{F}_{b\sqrt{t}}\Bigg]\Bigg]. \tag{5.21}$$

Using the inequalities

$$1-x\le e^{-x}\le 1-x+\frac{1}{2}x^2,\quad x>0, \tag{5.22}$$

for

$$x=\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})(1+o(1)) \tag{5.23}$$

we are able to bound (5.21) from below and above. First,

$$\mathbb{E}[x^2|\mathcal{F}_{b\sqrt{t}}]\le e^{-2(1+\sigma_1^2)b\sqrt{t}+2\sqrt{2}x_i(b\sqrt{t})-2\sqrt{2}y}\,\mathbb{E}\left[\left(Y^A_{b(t-\sqrt{t})}\right)^2\right](1+o(1))\,, \tag{5.24}$$

where $Y^A_{b(t-\sqrt{t})}$ is the truncated McKean martingale defined in (4.1). Note that its second moment is bounded by $D_2(r)$ (see (4.19)). Second, computing the conditional expectation given $\mathcal{F}_{b\sqrt{t}}$ yields, up to factors $1+o(1)$,

$$\begin{aligned}\mathbb{E}[x|\mathcal{F}_{b\sqrt{t}}]&=\mathbb{E}\Big[\sum_{\substack{1\le l\le n_l^{(i)}(b(t-\sqrt{t}))\\ x_i\in\mathcal{G}_{bt,A,\frac{1}{2}};x_l^{(i)}\in\mathcal{T}_{r,b(t-\sqrt{t})}}}C(a)E(x_i,x_l^{(i)})\Big|\mathcal{F}_{b\sqrt{t}}\Big]\\&\le e^{b(\sigma_1^2t-\sqrt{t})-\sqrt{2}y}\int_{K_t-A\sqrt{t}}^{K_t+A\sqrt{t}}e^{\sqrt{2}(z+x_i(b\sqrt{t}))-\frac{(z+x_i(b\sqrt{t})-\sqrt{2}\sigma_1^2bt)^2}{2\sigma_2^2(1-b)t}}\frac{e^{-z^2/2\sigma_1^2b(t-\sqrt{t})}dz}{\sqrt{2\pi\sigma_1^2b(t-\sqrt{t})}}\end{aligned} \tag{5.25}$$

where $K_t = \sqrt{2}tb\sigma_1^2 - x_i(b\sqrt{t})$. Performing the change of variables $z = w + K_t$ this is equal to

$$
\begin{aligned}
& e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\frac{1}{2}\log(1-b)-\sqrt{2}y} \int_{-A\sqrt{t}}^{A\sqrt{t}} e^{-\frac{w^2}{2\sigma_1^2 b(t-\sqrt{t})}-\frac{w^2}{2\sigma_2^2(1-b)t}} \frac{dw}{\sqrt{2\pi\sigma_1^2 b(t-\sqrt{t})}}(1+o(1)) \\
= \; & e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\frac{1}{2}\log(1-b)-\sqrt{2}y} \left(\frac{\sigma_2^2(1-b)}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2} \int_{-A\sqrt{t}}^{A\sqrt{t}} e^{-w^2/2t} \frac{dw}{\sqrt{2\pi t}}(1+o(1)) \\
= \; & e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y} \left(\frac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2} (1-\epsilon)(1+o(1)),
\end{aligned}
\tag{5.26}
$$

where $o(1) \leq O(t^{\gamma-1})$. Using Lemma 2.3 together with the independence of the Brownian bridge from its endpoint, we obtain that the right hand side of (5.26) multiplied by an additional factor $(1-\epsilon)$ is also a lower bound. Comparing this to (5.27), one sees that

$$
\frac{\mathbb{E}[x^2|\mathcal{F}_{b\sqrt{t}}]}{\mathbb{E}[x|\mathcal{F}_{b\sqrt{t}}]} \leq D_2(r)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})} \leq Ce^{-(1-\sigma_1^2)b\sqrt{t}+0(t^{\gamma/2})}, \tag{5.27}
$$

which tends to zero uniformly as $t \uparrow \infty$. Thus the second moment term is negligible. Hence we only have to control

$$
\begin{aligned}
& \mathbb{E}\left[\prod_{\substack{1\leq i\leq n(b\sqrt{t}) \\ x_i \in \mathcal{G}_{b\sqrt{t},B,\gamma}}} \left(1 - C(a)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y}\left(\frac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(-\sum_{\substack{1\leq i\leq n(b\sqrt{t}) \\ x_i \in \mathcal{G}_{b\sqrt{t},B,\gamma}}} C(a)e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})-\sqrt{2}y}\left(\frac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2}\right)(1+o(1))\right] \\
= \; & \mathbb{E}\left[\exp\left(-C(a)\left(\frac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2} e^{-\sqrt{2}y}\tilde{Y}^B_{b\sqrt{t},\gamma}\right)(1+o(1))\right]
\end{aligned}
\tag{5.28}
$$

where

$$
\tilde{Y}^B_{b\sqrt{t},\gamma} = \sum_{i=1}^{n(b\sqrt{t})} e^{-(1+\sigma_1^2)b\sqrt{t}+\sqrt{2}x_i(b\sqrt{t})} \mathbb{1}_{x_i(b\sqrt{t})-\sqrt{2}\sigma_1^2 b\sqrt{t}\in[-Bt^{\gamma/2},Bt^{\gamma/2}]}. \tag{5.29}
$$

Now from Lemma 4.6, $\tilde{Y}^B_{b\sqrt{t},\gamma}$ converges in probability and in L^1 to the random variable Y, when we let first t and then B tend to infinity. Since $Y^B_{b\sqrt{t},\gamma} \geq 0$ and $C(a) > 0$, it follows

$$
\begin{aligned}
& \lim_{B\uparrow\infty} \liminf_{t\uparrow\infty} \mathbb{E}\left[\exp\left(-C(a)\left(\frac{\sigma_2^2}{1-\sigma_1^2 b/\sqrt{t}}\right)^{1/2} \tilde{Y}^B_{b\sqrt{t},\gamma} e^{-\sqrt{2}y}\right)\right] \\
= \; & \lim_{B\uparrow\infty} \limsup_{t\uparrow\infty} \mathbb{E}\left[\exp\left(-\sigma_2 C(a)\tilde{Y}^B_{b\sqrt{t},\gamma} e^{-\sqrt{2}y}\right)\right] \\
= \; & \mathbb{E}\left[\exp\left(-\sigma_2 C(a) Y e^{-\sqrt{2}y}\right)\right].
\end{aligned}
\tag{5.30}
$$

Finally, letting r tend to $+\infty$, all the ϵ-errors (that are still present implicitly, vanish. This concludes the proof of Theorem 5.1. □

6 Existence of the limiting process

The following existence theorem is the basic step in the proof of Theorem 1.1.

Theorem 6.1. *Let $\sigma_1 < \sigma_2$. Then, the point processes $\mathcal{E}_t = \sum_{k\leq n(t)} \delta_{x_k(t)-\tilde{m}(t)}$ converges in law to a non-trivial point process $\mathcal{E}$.*

Proof. It suffices to show that, for $\phi \in \mathcal{C}_c(\mathbb{R})$ positive, the Laplace functional

$$\Psi_t(\phi) = \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\right], \tag{6.1}$$

of the processes $\mathcal{E}_t$ converges. First observe that this limit cannot be zero, since the maximum of the time inhomogeneous BBM converges by Theorem 5.1. As for standard BBM (see e.g. [3]), it follows

$$\lim_{N\to\infty}\lim_{t\to\infty} \mathbb{P}\left[\mathcal{E}_t(B) > N\right] = 0, \text{ for any bounded } B \subset \mathbb{R}, \tag{6.2}$$

which implies the locally finiteness of the limiting point process. As in [3] we decompose

$$\Psi_t(\phi) = \Psi_t^{<\delta}(\phi) + \Psi_t^{>\delta}(\phi), \tag{6.3}$$

where

$$\begin{aligned}\Psi_t^{<\delta}(\phi) &= \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\mathbb{1}_{\max\mathcal{E}_t\leq\delta}\right]\\ \Psi_t^{>\delta}(\phi) &= \mathbb{E}\left[\exp\left(-\int \phi(y)\mathcal{E}_t(dy)\right)\mathbb{1}_{\max\mathcal{E}_t>\delta}\right].\end{aligned} \tag{6.4}$$

Here we write shorthand $\max\mathcal{E}_t \leq \delta$ for $\max_{k\leq n(t)}(x_k(t) - m(t) \leq \delta$. By Theorem 5.1 we have

$$\limsup_{\delta\to\infty}\limsup_{t\to\infty}\Psi_t^{>\delta}(\phi) \leq \limsup_{\delta\to\infty}\limsup_{t\to\infty}\mathbb{P}[\max\mathcal{E}_t > \delta] = 0. \tag{6.5}$$

Hence it remains to analyse the behaviour of $\Psi_t^{<\delta}(\phi)$. We claim that

$$\lim_{\delta\to\infty}\lim_{t\to\infty}\Psi_t^{<\delta}(\phi) = \Psi(\phi) \tag{6.6}$$

exists and is strictly smaller than 1. To see this set

$$\bar{\phi}(z) = \phi(\sigma_2 z) \tag{6.7}$$

and

$$g_\delta(z) = e^{-\bar{\phi}(-z)}\mathbb{1}_{\{-z\sigma_2\leq\delta\}}. \tag{6.8}$$

Moreover, define

$$u_\delta(t,z) = 1 - \mathbb{E}\left[\prod_{j\leq n(t)} g_\delta(z - \bar{x}_j(t))\right]. \tag{6.9}$$

where $\{\bar{x}_j(t), 1 \leq j \leq n(t)\}$ are the particles of a standard BBM with variance 1. We observe that by [18] $u_\delta(t,x)$ solves the F-KPP equation (3.2) with initial condition $u_\delta(0,x) = 1 - g_\delta(x)$. Next we verify Assumptions (i)-(iv) of Proposition 3.1. (i) is clear. Moreover, $g_\delta(x) = 1$ for x large enough in the positive , and $g_\delta(x) = 0$ for $-x$ large enough, so that Conditions (ii)-(iv) of Proposition 3.1 are satisfied. Now

$$\begin{aligned}\Psi_t^{<\delta}(\phi) &= \mathbb{E}\left[\prod_{i\leq n(bt)}\mathbb{E}\left[\prod_{x_j^i\leq n_i((1-b)t)} g_\delta((\tilde{m}(t) - x_i(bt) - x_j^i((1-b)t))/\sigma_2)\middle|\mathcal{F}_{bt}\right]\right]\\ &= \mathbb{E}\left[\prod_{i\leq n(bt)}\mathbb{E}\left[\prod_{\bar{x}_j^i\leq n_i((1-b)t)} g_\delta((\tilde{m}(t) - x_i(bt))/\sigma_2 - \bar{x}_j^i((1-b)t))\middle|\mathcal{F}_{bt}\right]\right],\end{aligned} \tag{6.10}$$

where for each i, $\bar{x}^i_j$ are the particles of iid standard BBMs. By Proposition 3.1 and the same calculations as in the proof Theorem 5.1 we have that this converges, as $t \to \infty$, to

$$\mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi,\delta)Y\right)\right], \tag{6.11}$$

where $C(a,\phi,\delta)$ is the constant that appears in Lemma 3.2, with initial condition $g_\delta(z)$, i.e.

$$C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2}(1-e^{-2za})dz, \tag{6.12}$$

where $a = \sqrt{2}(\sigma_2 - 1)$ and u_δ is the solution to the F-KPP equation (3.2) with initial condition $u_\delta(0,z) = 1 - e^{-\bar{\phi}(z)}\mathbb{1}_{\{z\sigma_2 \le \delta\}}$. Thus the limit $\lim_{t\to\infty} \Psi_t^{<\delta}(\phi) = \Psi^{<\delta}(\phi)$ exists. The limit $\delta \uparrow \infty$ then exists by the same argument as in the proof of Theorem 3.1 of [3]: the function

$$\delta \to \Psi^{<\delta}(\phi) \tag{6.13}$$

is increasing and bounded, Moreover, the maximum is an atom of $\mathcal{E}_t$ and ϕ is nonnegative, and so

$$\Psi_t^{<\delta}(\phi) \le \mathbb{E}\left[\exp\left(-\phi(\max \mathcal{E}_t)\right) \mathbb{1}_{\{\max \mathcal{E}_t \le \delta\}}\right] \tag{6.14}$$

The limit as $t \to \infty$ and $\delta \to \infty$ of the right hand side of (6.14) exists by Theorem 5.1. Hence

$$\Psi(\phi) = \lim_{\delta\to\infty} \Psi^\delta(\phi) < 1, \tag{6.15}$$

by monotone convergence. This implies the existence of the limiting process. □

Proposition 6.2. *Let $v(t,x)$, $v_\delta(t,x)$ be solutions of the F-KPP equation with initial data $v(0,x) = 1 - e^{-\bar{\phi}(-x)}$ and $v_\delta(0,x) = 1 - e^{-\bar{\phi}(-x)}\mathbb{1}_{\{-x\sigma_2 \le \delta\}}$ respectively. Set*

$$C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2}\left(1-e^{-2az}\right)dz \tag{6.16}$$

Then $\lim_{\delta\to\infty} C(a,\phi,\delta)$ exists and is given by

$$C(a,\phi) = \lim_{\delta\to\infty} C(a,\phi,\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.17}$$

Moreover,

$$\lim_{t\to\infty} \Psi_t(\phi) = \mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi)Y\right)\right]. \tag{6.18}$$

Proof. First we note that

$$C(a,\phi.\delta) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.19}$$

To see this, note that for any $K < \infty$,

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^K v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz \le \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} K e^{-a^2 t/2} e^{(\sqrt{2}+a)K} = 0. \tag{6.20}$$

Obviously,

$$C(a,\phi,\delta) \le \liminf_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.21}$$

Due to (6.20), for any $K < \infty$,

$$\begin{aligned} &C(a,\phi,\delta) - \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz \\ &\ge -e^{-aK} \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \end{aligned} \tag{6.22}$$

Since this holds for all K, and since the finiteness of the limsup in (6.22) follows from the finiteness of $C(a,\phi,\delta)$, we also have that

$$C(a,\phi,\delta) \geq \limsup_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty v_\delta(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz, \tag{6.23}$$

and Eq. (6.19) follows. It remains to control the limit as $\delta \uparrow \infty$ of the right-hand side of (6.19). But an exact rerun of the proof of Lemma 4.10 in [3] using Lemma 6.4 below instead of Lemma 4.8 of [3] yields that

$$\lim_{\delta\uparrow\infty} \lim_{t\uparrow\infty} \int_0^\infty v_\delta(t, x+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz \equiv \lim_{\delta\uparrow\infty} F(\delta) \equiv F \tag{6.24}$$

exists. By (6.11) and (6.24) we have

$$\lim_{t\to\infty} \Psi_t^{<\delta}(\phi) = \mathbb{E}\left[\exp\left(-\sigma_2 C(a,\phi,\delta) Y\right)\right] = \mathbb{E}\left[\exp\left(-\frac{\sigma_2}{\sqrt{2\pi}} F(\delta) Y\right)\right]. \tag{6.25}$$

This converges for $\delta \to \infty$ to

$$\mathbb{E}\left[\exp\left(-\frac{\sigma_2}{\sqrt{2\pi}} F Y\right)\right]. \tag{6.26}$$

Hence $F = 0$ would imply

$$\lim_{\delta\to\infty} \lim_{t\to\infty} \Psi_t(\phi) = 1, \tag{6.27}$$

which contradicts (6.15) and Theorem 6.1. Hence $F > 0$. Moreover, (6.26) implies (6.18), which concludes the proof of Proposition 6.2. $\square$

We recall the following estimate for the tail probabilities of standard BBM.

Lemma 6.3 ([2], Corollary 10)**.** *There exists $t_0 < \infty$, such that for $z > 1$ and $t \geq t_0$*

$$\mathbb{P}\left[\max_{k\leq n(t)} \bar{x}_k(t) - \sqrt{2}t + \frac{3}{2\sqrt{2}} \log t \geq z\right] \leq \rho z \exp\left(-\sqrt{2}z - \frac{z^2}{2t} + \frac{3z}{2\sqrt{2}} \frac{\log t}{t}\right), \tag{6.28}$$

for some constant $\rho > 0$.

Lemma 6.4. *Let u be a solution of the F-KPP equation with initial data satisfying Assumptions (i)-(iv) of Proposition 3.1. Let*

$$C(a) = \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t, z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz. \tag{6.29}$$

Then for any $x \in \mathbb{R}$:

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_0^\infty u(t, x+z+\sqrt{2}t) e^{(\sqrt{2}+a)z - a^2 t/2} dz = C(a) e^{-(\sqrt{2}+a)x}. \tag{6.30}$$

Moreover, for any bounded continuous function $h(x)$, that is zero for x small enough

$$\begin{aligned} &\lim_{t\to\infty} \int_{-\infty}^0 \mathbb{E}\left[h\left(y + \max \bar{x}_i(t) - \sqrt{2}t\right)\right] \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{2}+a)y - a^2 t/2} dy \\ = \quad & C(a) \int_{\mathbb{R}} h(z)(\sqrt{2}+a) e^{-(\sqrt{2}+a)z} dz, \end{aligned} \tag{6.31}$$

where $\{\bar{x}_i(t), i \leq n(t)\}$ are the particles of a standard BBM with variance 1. Here $C(a)$ is the constant from (6.29) for u satisfying the initial condition $\mathbb{1}_{\{x\leq 0\}}$.

Proof. We have by a simple change of variables

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}}\int_0^\infty u(t,z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz \\ = \quad &\frac{e^{(\sqrt{2}+a)x}}{\sqrt{2\pi}}\int_{-x}^\infty u(t,x+z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz. \end{aligned} \tag{6.32}$$

Moreover, $\lim_{t\to\infty} u(t,x+z+\sqrt{2}t)=0$ implies

$$\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-x}^0 u(t,x+z+\sqrt{2}t)e^{(\sqrt{2}+a)z-a^2t/2}dz=0, \tag{6.33}$$

which proves (6.30). Moreover, (6.30) with initial condition $\mathbb{1}_{\{x\le 0\}}$ implies that (6.31) holds for $h(x)=\mathbb{1}_{[b,\infty)}, b\in\mathbb{R}$. For general h (6.31) follows in the same way as Lemma 4.11 in [3] by linearity and a monotone class argument. □

7 The auxiliary process

We define the following auxiliary process that has the same limiting behaviour as that of the two-speed BBM. We will denote the law of these processes by P and expectations by E. If desired, all ingredients of the auxiliary process can be thought of to be defined on a new probability space. Let $(\eta_i; i\in\mathbb{N})$ be the atoms of a Poisson point process η on $(-\infty,0)$ with intensity measure

$$\tfrac{\sigma_2}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z}e^{-a^2t/2}dz. \tag{7.1}$$

For each $i\in\mathbb{N}$ consider independent standard BBMs $\bar{x}^i$. The auxiliary point process of interest is the superposition of the i.i.d BBMs with drift shifted by $\eta_i+\frac{1}{\sqrt{2}+a}\log Y$, where a is the constant defined in (5.12):

$$\Pi_t=\sum_{i,k}\delta_{\left(\eta_i+\frac{1}{\sqrt{2}+a}\log Y+\bar{x}^i_k(t)-\sqrt{2}t\right)\sigma_2}. \tag{7.2}$$

Remark 7.1. *The form of the auxiliary process is similar to the case of standard BBM, but with a different intensity of the Poisson process. In particular, the intensity decays exponentially with t. This is a consequence of the fact that particles at the time of the speed change were forced to be $O(t)$ below the line $\sqrt{2}t$, in contrast to the $O(\sqrt{t})$ in the case of ordinary BBM. The reduction of the intensity of the process with t forces the particles to be selected at these locations.*

Theorem 7.2. *Let $\mathcal{E}_t$ be the extremal process of the two-speed BBM. Then*

$$\lim_{t\to\infty}\mathcal{E}_t \overset{law}{=} \lim_{t\to\infty}\Pi_t. \tag{7.3}$$

Proof. Using the notation $\bar{\phi}(z)=\phi(\sigma_2 z)$ and by the form of the Laplace functional of a Poisson point process we have

$$\begin{aligned} &E\left[\exp\left(-\int\phi(x)\Pi_t(dx)\right)\right] \\ = \quad &E\left[\exp\left(-\sigma_2\int_{-\infty}^0\left\{1-E\left[\exp\left(-\sum_{k\le n(t)}\bar{\phi}\left(z+\bar{x}_k(t)-\sqrt{2}t+\frac{\log Y}{\sqrt{2}+a}\right)\right)\right]\right\}\right.\right. \\ &\qquad\left.\left.\times e^{-(\sqrt{2}+a)z}e^{-a^2t/2}dz\right)\right] \\ = \quad &E\left[\exp\left(\frac{\sigma_2}{\sqrt{2\pi}}\int_0^\infty u\left(t,z+\sqrt{2}t-\frac{1}{\sqrt{2}+a}\log Y\right)e^{(\sqrt{2}+a)z}e^{-a^2t/2}dz\right)\right], \end{aligned} \tag{7.4}$$

with

$$u(t,x) = 1 - E\left[\exp\left(-\sum_{k\le n(t)} \bar\phi(-x+\bar x_k(t))\right)\right]. \tag{7.5}$$

By Lemma 6.4 we have

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}\int_0^\infty u\left(t, z+\sqrt{2}t-\frac{1}{\sqrt{2}+a}\log Y\right)e^{(\sqrt{2}+a)z}e^{-a^2t/2}dz\\ =\;& Y\lim_{t\to\infty}\frac{1}{\sqrt{2\pi}}\int_0^\infty u(t,z+\sqrt{2}t)e^{(\sqrt{2}+a)z}e^{-a^2t/2}dz,\end{aligned} \tag{7.6}$$

which exists and is strictly positive by Proposition 6.2. This implies that the Laplace functionals of $\lim_{t\to\infty}\Pi_t$ and of the extremal process of the two-speed BBM are equal. □

The following proposition shows that in spite of the different Poisson ingredients, when we look at the process of the extremes of each of the $x^i(t)$, we end up with a Poisson point process just like in the standard BBM case.

Proposition 7.3. *Define the point process*

$$\Pi_t^{ext} \equiv \sum_i \delta_{\left(\eta_i+\frac{1}{\sqrt{2}+a}\log Y+\max_{k\le n_i(t)}\bar x_k^i(t)-\sqrt{2}t\right)\sigma_2}. \tag{7.7}$$

Then

$$\lim_{t\to\infty}\Pi_t^{ext} \stackrel{law}{=} P_Y \equiv \sum_{i\in\mathbb{N}}\delta_{p_i}, \tag{7.8}$$

where P_Y is the Poisson point process on $\mathbb{R}$ with intensity measure $\sigma_2 C(a) Y\sqrt{2}e^{-\sqrt{2}x}dx$.

Proof. We consider the Laplace functional of Π_t^{ext}. Let $M^{(i)}(t) = \max\bar x_k^{(i)}(t)$ and as before $\bar\phi(z) = \phi(\sigma_2 z)$. We want to show

$$\begin{aligned}&\lim_{t\uparrow\infty}E\left[\exp\left(-\sum_i\bar\phi(\eta_i+M^{(i)}(t)-\sqrt{2}t\right)\right]\\ =\;&\exp\left(-\sigma_2C(a)\int_{-\infty}^\infty\left(1-e^{-\phi(x)}\right)\sqrt{2}e^{-\sqrt{2}x}dx\right).\end{aligned} \tag{7.9}$$

Since η_i is a Poisson point process and the $M^{(i)}$ are i.i.d. we have

$$\begin{aligned}&E\left[\exp\left(-\sum_i\bar\phi(\eta_i+M^{(i)}(t)-\sqrt{2}t\right)\right]\\ =\;&\exp\left(-\sigma_2\int_{-\infty}^0 E\left[1-e^{-\bar\phi(z+M(t)-\sqrt{2}t)}\right]e^{-(\sqrt{2}+a)z-a^2t/2}\frac{dz}{\sqrt{2\pi}}\right),\end{aligned} \tag{7.10}$$

where $M(t)$ has the same distribution as one the variables $M^{(i)}(t)$. Now we apply Lemma 6.4 with $h(x) = 1-e^{-\bar\phi(z)}$. Hence the result follows by using that $\bar\phi(z)=\phi(\sigma_2 z)$ and $\sqrt{2}+a=\sqrt{2}\sigma_2$ together with the change of variables $x=\sigma_2 z$. □

The following proposition states that the Poisson points of the auxiliary process contribute to the limiting process come from a neighbourhood of $-at$.

Proposition 7.4. *Let $z\in\mathbb{R}, \epsilon>0$. Let η_i be the atoms of a Poisson point process with intensity measure $Ce^{-(\sqrt{2}+a)x-a^2t/2}dx$ on $(-\infty,0]$. Then there exists $B<\infty$ such that*

$$\sup_{t\ge t_0}P\left(\exists i,k:\eta_i+\bar x_k^i(t)-\sqrt{2}t\ge z, \eta_i\notin[-at-B\sqrt{t},-at+B\sqrt{t}]\right)\le\epsilon. \tag{7.11}$$

Proof. By a first order Chebychev inequality we have

$$\begin{aligned}
& P\left(\exists i,k:\eta_i+\bar{x}_k^{(i)}(t)-\sqrt{2}t\geq z,\eta_i>-at+B\sqrt{t}\right) \\
\leq\ & C\int_{-at+B\sqrt{t}}^{0}P\left(\max\bar{x}_k(t)\geq\sqrt{2}t-x+z\right)e^{-(\sqrt{2}+a)x}e^{-a^2t/2}dx \\
=\ & C\int_{0}^{at-B\sqrt{t}}P\left(\max\bar{x}_k(t)\geq\sqrt{2}t+x+z\right)e^{(\sqrt{2}+a)x}e^{-a^2t/2}dx, \qquad (7.12)
\end{aligned}$$

by the change of variables $x\to -x$. Using the asymptotics of Lemma 6.3 we can bound (7.12) from above by

$$\begin{aligned}
& \rho C\int_0^{at-B\sqrt{t}}t^{-1/2}e^{-\sqrt{2}(x+z)}e^{-(x+z)^2/2t}e^{(\sqrt{2}+a)x}e^{-a^2t/2}dx \\
\leq\ & \rho C\int_{-a\sqrt{t}}^{-B}e^{z^2/2}dz(1+o(1)), \qquad (7.13)
\end{aligned}$$

by changing variables $x\to x/\sqrt{t}-a\sqrt{t}$. This is a Gaussian integral and can be made as small as we want by choosing B large enough. Similarly one bounds

$$P\left(\exists i,k:\eta_i+x_k^i(t)-\sqrt{2}t\geq z,\eta_i<-at-B\sqrt{t}\right)\leq C\rho\int_B^{\infty}e^{z^2/2}dz(1+o(1)). \qquad (7.14)$$

This concludes the proof. □

The next proposition describes the law of the clusters $\bar{x}_k^{(i)}$. This is analogous to Theorem 3.4 in [3].

Proposition 7.5. *Let $x=at+o(t)$ and $\{\tilde{x}_k(t),k\leq n(t)\}$ be a standard BBM under the conditional law $P\left(\cdot\middle|\{\max\tilde{x}_k(t)-\sqrt{2}t-x>0\}\right)$. Then the point process*

$$\sum_{k\leq n(t)}\delta_{\tilde{x}_k(t)-\sqrt{2}t-x} \qquad (7.15)$$

converges in law under $P\left(\cdot\middle|\{\max\tilde{x}_k(t)-\sqrt{2}t-x>0\}\right)$ as $t\to\infty$ to a well defined point process $\bar{\mathcal{E}}$. The limit does not depend on $x-at$ and the maximum of $\bar{\mathcal{E}}$ shifted by x has the law of an exponential random variable with parameter $\sqrt{2}+a$.

Proof. Set $\bar{\mathcal{E}}_t=\sum_k\delta_{\tilde{x}_k(t)-\sqrt{2}t}$ and $\max\bar{\mathcal{E}}_t=\max\tilde{x}_k(t)-\sqrt{2}t$. First we show that for $X>0$

$$\lim_{t\to\infty}P\left(\max\bar{\mathcal{E}}_t>X+x\middle|\max\bar{\mathcal{E}}_t>x\right)=e^{-(\sqrt{2}+a)X}. \qquad (7.16)$$

To see this we rewrite the conditional probability as $\frac{P\left[\max\bar{\mathcal{E}}_t>X+x\right]}{P\left[\max\bar{\mathcal{E}}_t>x\right]}$ and use the uniform bounds of Proposition 4.3 in [3]. Observing that

$$\lim_{t\to\infty}\frac{\Psi(r,t,X+x+\sqrt{2}t)}{\Psi(r,t,x+\sqrt{2}t)}=e^{-(\sqrt{2}+a)X}, \qquad (7.17)$$

where Ψ is defined in Equation (3.4), we get (7.16) by first taking $t\to\infty$ and then $r\to\infty$. The general claim of Proposition 7.5 follows in exactly the same way from (7.16) as Theorem 3.4. in [3]. □

Define the gap process

$$D_t=\sum_k\delta_{\tilde{x}_k(t)-\max_j\tilde{x}_j(t)}. \qquad (7.18)$$

Denote by ξ_i the atoms of the limiting process $\bar{\mathcal{E}}$, i.e. $\bar{\mathcal{E}} \equiv \sum_j \delta_{\xi_j}$ and define

$$D \equiv \sum_j \delta_{\Lambda_j}, \quad \Lambda_j = \xi_j - \max_i \xi_i. \tag{7.19}$$

D is a point process on $(-\infty.0]$ with an atom at 0.

Corollary 7.6. *Let $x = -at + o(t)$. In the limit $t \to \infty$ the random variables D_t and $x + \max \bar{\mathcal{E}}_t$ are conditionally independent on the event $\{x + \max \bar{\mathcal{E}}_t > b\}$ for any $b \in \mathbb{R}$. More precisely, for any bounded function f, h and $\bar{\phi} \in C_c(\mathbb{R})$,*

$$\begin{aligned} &\lim_{t\to\infty} E\left[f\left(\int \bar{\phi}(z)D_t(dz)\right) h(x + \max \bar{\mathcal{E}}) \big| x + \max \bar{\mathcal{E}} > b\right] \\ = \quad & E\left[f\left(\int \bar{\phi}(z)D(dz)\right)\right] \frac{\int_b^\infty h(z)(\sqrt{2}+a)e^{-(\sqrt{2}+a)z}dz}{e^{-(\sqrt{2}+a)b}}. \end{aligned} \tag{7.20}$$

Proof. The proof is essentially identical to the proof of Corollary 4.12 in [3]. Let us outline, for the benefit of the readers, the structure of the proof. First, by Proposition 7.5 the pair $(\bar{\mathcal{E}}_t, \max \bar{\mathcal{E}}_t - x)$, converge under the law conditioned on $\max \bar{\mathcal{E}}_t - x > 0$ to $(\mathcal{E}, e)$, where e is an exponential random variable with parameter $\sqrt{2} + a$ and $\mathcal{E}$ is independent of the precise value of the conditioning. A general continuity lemma, stated and proven as Lemma 4.13 in [3], shows that this implies the convergence of the processes $(D_t, \max \bar{\mathcal{E}}_t - x)$ to a pair $(\mathcal{D}, e)$ where D_t is given in (7.18) is related to $\bar{\mathcal{E}}_t$ by a random shift of its atoms. The fact that $\mathcal{D}$ and e are independent follows from an explicit computation, just as in the proof of Corollary 4.12 in [3]. We do not repeat the details. □

Finally we come to the description of the extremal process as seen from the Poisson process of cluster extremes, which is the formulation of Theorem 1.1.

Theorem 7.7. *Let P_Y be as in (7.8) and let $\{D^{(i)}, i \in \mathbb{N}\}$ be a family of independent copies of the gap-process (7.19) with atoms $\Lambda_j^{(i)}$. Then the point process $\mathcal{E}_t$ converges in law as $t \to \infty$ to a Poisson cluster point process $\mathcal{E}$ given by*

$$\mathcal{E} \overset{law}{=} \sum_{i,j} \delta_{p_i + \sigma_2 \Lambda_j^{(i)}}. \tag{7.21}$$

Proof. Also this proof is now very close to that of Theorem 2.1 in [3]. First note that the Laplace functional of the process $\mathcal{E}$ is given by

$$\begin{aligned} &E\left[\exp\left(-\int \phi(x)\mathcal{E}(dx)\right)\right] \\ &= E\left[\exp\left(-\sigma_2 C(a) Y \int_{\mathbb{R}} E\left[1 - e^{-\int \phi(y+x)D(dx)}\right] \sqrt{2}e^{-\sqrt{2}y} dy\right)\right]. \end{aligned} \tag{7.22}$$

Thus, by Theorem 7.2, we have to show that the Laplace functional of the processes Π_t converge to this expression. In the proof of that theorem, we have seen that

$$\begin{aligned} &\lim_{t\uparrow\infty} E\left[\exp\left(-\int \phi(x)\Pi_t(dx)\right)\right] \\ &= E\left[\exp\left(-\sigma_2 Y \lim_{t\uparrow\infty} \int_{-\infty}^0 E\left[1 - \exp\left(-\int \bar{\phi}(z+x)\bar{\mathcal{E}}_t(dx)\right)\right] \frac{e^{-(\sqrt{2}+a)z - a^2 t/2}}{\sqrt{2\pi}} dz\right)\right]. \end{aligned} \tag{7.23}$$

We rewrite

$$\begin{aligned}&\int_{-\infty}^{0} E\left[1-\exp\left(-\int \bar\phi(z+x)\bar{\mathcal{E}}_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz\\ =\quad &\int_{-\infty}^{0} E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz, \qquad (7.24)\end{aligned}$$

where $f(x)=1-e^{-x}$, $T_z\bar\phi(x)=\bar\phi(z+x)$, $f(0)=0$. Using the localisation estimate of Proposition 7.4 we have that (7.24) is equal to

$$\Omega_t(B)+\int_{-at-B\sqrt{t}}^{-at+B\sqrt{t}} E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right]\frac{1}{\sqrt{2\pi}}e^{-(\sqrt{2}+a)z-a^2t/2}dz, \qquad (7.25)$$

where $\lim_{B\to\infty}\sup_{t\geq t_0}\Omega_t(B)=0$. Let $m_{\bar\phi}$ be the minimum of the support of $\bar\phi$. we observe that

$$f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)=0, \qquad (7.26)$$

when $z+\max\bar{\mathcal{E}}_t<m_{\bar\phi}$. Moreover, $P\left[z+\max\bar{\mathcal{E}}_t=m_{\bar\phi}\right]=0$. Hence

$$\begin{aligned}&E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\right] \qquad (7.27)\\ =\quad &E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\mathbb{1}_{\{z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\}}\right]\\ =\quad &E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\big|z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]P\left[z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right].\end{aligned}$$

Now by Corollary 7.6, for z in the range of integration in (7.25), on the event we are conditioning on in (7.27), the random variables D_t and $\max\bar{\mathcal{E}}_t+z-m_{\bar\phi}$ converge to independent random variables (D,e), where e is exponential with parameter $\sqrt{2}+a$. Hence

$$\begin{aligned}&\lim_{t\uparrow\infty} E\left[f\left(\int\left\{T_{z+\max\bar{\mathcal{E}}_t}\bar\phi(x)\right\}D_t(dx)\right)\big|z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]\\ &=\int_0^\infty(\sqrt{2}+a)e^{-(\sqrt{2}+a)u}E\left[f\left(\int\bar\phi(u+m_{\bar\phi}+x)D(dx)\right)\right]du \qquad (7.28)\\ &=\int_{m_{\bar\phi}}^\infty(\sqrt{2}+a)e^{-(\sqrt{2}+a)(u-m_{\bar\phi})}E\left[f\left(\int\bar\phi(u+x)D(dx)\right)\right]du.\end{aligned}$$

Note that this expression is independent of z. Thus it remains to compute the integral of $P\left[z+\max\bar{\mathcal{E}}_t>m_{\bar\phi}\right]$. But this converges to $e^{-(\sqrt{2}+a)m_{\bar\phi}}$ by (6.30) in Lemma 6.4, together with the localisation estimates of Proposition 7.4 (which this time allows to re-extend the range of integration). Putting this together with (7.28) and changing variables $x=\sigma_2 z$ shows that the right-hand side of (7.23) is indeed equal to the right-hand side of (7.22). This proves the theorem. □

8 The case $\sigma_1>\sigma_2$

In this section we proof Theorem 1.3. The existence of the process $\mathcal{E}$ from (1.15) will be a byproduct of the proof.

The following result is contained in the calculation of the maximal displacement in [10].

Lemma 8.1. *([10]) For all $\epsilon > 0, d \in \mathbb{R}$ there exists a constant D large enough such that for t sufficiently large*

$$\mathbb{P}\left[\exists k \leq n(t) : x_k(t) > m(t) + d \text{ and } x_k(bt) < m_1(bt) - D\right] < \epsilon. \tag{8.1}$$

Proof of Theorem 1.3. First we establish the existence of a limiting process. Note that $m(t) = m_1(bt) + m_2((1-b)t)$, where $m_i(s) = \sqrt{2}\sigma_i s - \frac{3}{2\sqrt{2}}\sigma_i \log s$. Recall

$$\bar{\phi}(z) = \phi(\sigma_2 z) \tag{8.2}$$

and

$$g_\delta(z) = e^{-\bar{\phi}(-z)} \mathbb{1}_{\{-z \leq \delta\}}. \tag{8.3}$$

Using that the maximal displacement is $m(t)$ in this case we can proceed as in the proof of Theorem 6.1 up to (6.9) and only have to control

$$\Psi_t^{<\delta}(\phi) = \mathbb{E}\left[\prod_{i \leq n(tb)} \mathbb{E}\left[\prod_{j \leq n_i((1-b)t)} g_\delta((m(t) - x_i(bt))/\sigma_2 - \bar{x}_j^i((1-b)t)) \big| \mathcal{F}_{tb}\right]\right], \tag{8.4}$$

where $\bar{x}_j^i((1-b)t)$ are the particles of a standard BBM at time $(1-b)t$ and $x_i(bt)$ are the particles of a BBM with variance σ_1 at time bt. Using Lemma 8.1 and Theorem 1.2 of [10] as in the proof of Theorem 5.1 above, we obtain that (8.4), for t sufficiently large, equals

$$E\left[\prod_{\substack{i \leq n(bt) \\ x_i(bt) > m_1(bt) - D}} \mathbb{E}\left[\prod_{j \leq n_i((1-b)t)} g_\delta\left(\tfrac{(m(t) - x_i(bt))}{\sigma_2} - \bar{x}_j^i((1-b)t)\right) \big| \mathcal{F}_{tb}\right]\right] + O(\epsilon). \tag{8.5}$$

The rest of the proof has an iterated structure. In a first step we show that conditioned on $\mathcal{F}_{bt}$ for each $i \leq n(bt)$ the points $\{x_i(bt) + x_j^i((1-b)t) - m(t) | x_i(bt) > m_1(bt) - D\}$ converge to the corresponding points of the point process $x_i(bt) - m_1(bt) + \sigma_2 \tilde{\mathcal{E}}^{(i)}$, where $\tilde{\mathcal{E}}^{(i)}$ are independent copies of the extremal process (1.6) of standard BBM. To this end observe that

$$u_\delta((1-b)t, z) = 1 - \mathbb{E}\left[\prod_{j \leq n((1-b)t)} g_\delta(z - \bar{x}_j^i((1-b)t))\right] \tag{8.6}$$

solves the F-KPP equation (3.2) with initial condition $u_\delta(0,z) = 1 - e^{-\bar{\phi}(-z)} \mathbb{1}_{\{-z \leq \delta\}}$. Moreover, the assumptions of Lemma 4.9 in [3] are satisfied. Hence (8.5) is equal to

$$\epsilon + \mathbb{E}\left[\prod_{\substack{i \leq n(bt) \\ x_i(bt) > m_1(bt) - D}} \left(\mathbb{E}\left[e^{-C(\bar{\phi},\delta) Z e^{-\sqrt{2}\frac{m_1(bt) - x_i(bt)}{\sigma_2}}} \big| \mathcal{F}_{bt}\right] (1 + o(1))\right)\right]. \tag{8.7}$$

Here $C(\bar{\phi}, \delta)$ is from standard BBM, i.e.

$$C(\bar{\phi}, \delta) = \lim_{t \uparrow \infty} \sqrt{\tfrac{2}{\pi}} \int_0^\infty u_\delta(t, y + \sqrt{2}t) y e^{\sqrt{2}y} dy, \tag{8.8}$$

see Eq. 4.49 in [3]. Note furthermore that already in (8.7) the concatenated structure of the limiting point process becomes visible. In a second step we establish that the points $x_i(bt) - m_1(t)$ that have a descendant in the lead at time t converge to $\widetilde{\mathcal{E}}$.

Define

$$h_{\delta,D}(y) \equiv \begin{cases} \mathbb{E}\left[\exp\left(-C(\bar{\phi},\delta)Ze^{-\sqrt{2}\frac{\sigma_1}{\sigma_2}y}\right)\right], & \text{if } \sigma_1 y < D, \\ 1, & \text{if } \sigma_1 y \geq D. \end{cases} \tag{8.9}$$

Then the expectation in (8.7) can be written as (we ignore the error term $o(1)$ which is easily controlled using that the probability that the number of terms in the product is larger than N tends to zero as $N \uparrow \infty$, uniformly in t)

$$\mathbb{E}\left[\prod_{i\leq n(bt)} h_{\delta,D}(m_1(bt)/\sigma_1 - \bar{x}_i(t))\right], \tag{8.10}$$

where now $\bar{x}$ is standard BBM. Defining

$$v_{\delta,D}(t,z) = 1 - \mathbb{E}\left[\prod_{i\leq n(t)} h_{\delta,D}(z - \bar{x}_i(bt))\right], \tag{8.11}$$

$v_{\delta,D}$ is a solution of the F-KPP equation (3.2) with initial condition $v_{\delta,D}(0,z) = 1-h_{\delta,D}(z)$. But this initial condition satisfies the assumptions of Bramson's Theorem A in [6] and therefore,

$$v_{\delta,D}(t,m(t)+x) \to \mathbb{E}\left[e^{-\tilde{C}(D,Z,C(\bar{\phi},\delta))\tilde{Z}e^{-\sqrt{2}x}}\right]. \tag{8.12}$$

where $\tilde{Z}$ is an independent copy of Z and

$$\tilde{C}(D,Z,C(\bar{\phi},\delta)) = \lim_{t\uparrow\infty}\sqrt{\tfrac{2}{\pi}}\int_0^\infty v_{\delta,D}(t,y+\sqrt{2}t)ye^{\sqrt{2}y}dy. \tag{8.13}$$

By the same argumentation as in standard BBM setting (see [3]) one obtains that

$$\tilde{C}(Z,C(\bar{\phi},\delta)) \equiv \lim_{D\uparrow\infty}\tilde{C}(D,Z,C(\bar{\phi},\delta)) = \lim_{t\uparrow\infty}\sqrt{\tfrac{2}{\pi}}\int_0^\infty v_\delta(t,y+\sqrt{2}t)ye^{\sqrt{2}y}dy, \tag{8.14}$$

where v_δ is the solution of the F-KPP equation with initial condition $v(0,z) = 1 - h_\delta(z)$ with

$$h_\delta(z) = \mathbb{E}\left[\exp\left(-C(\bar{\phi},\delta)Ze^{-\sqrt{2}\frac{\sigma_1}{\sigma_2}z}\right)\right]. \tag{8.15}$$

The next step is to take the limit $\delta \to \infty$. Using Lemma 4.10 of [3] we have that $C(\bar{\phi},\delta)$ is monotone decreasing in δ and $\lim_{\delta\to\infty} C(\bar{\phi},\delta) = C(\bar{\phi})$, exists and is strictly positive, where

$$C(\bar{\phi}) = \lim_{t\uparrow\infty}\sqrt{\tfrac{2}{\pi}}\int_0^\infty u(t,y+\sqrt{2}t)ye^{\sqrt{2}y}dy. \tag{8.16}$$

Here $u(t,x)$ is a solution to the F-KPP equation (3.2) with initial condition $u(0,x) = 1 - e^{-\bar{\phi}(-x)}$. Using the same monotonicity arguments shows that also

$$\lim_{\delta\to\infty}\tilde{C}(Z,C(\bar{\phi},\delta)) = \tilde{C}(Z,C(\bar{\phi})). \tag{8.17}$$

Therefore, taking the limit first as $D\uparrow\infty$ and then $\delta\uparrow\infty$ in the left-hand side of (8.12), we get that

$$\begin{aligned} \lim_{t\to\infty}\Psi_t(\phi(\cdot+x)) &= \lim_{\delta\uparrow\infty}\lim_{t\to\infty}\Psi_t^{<\delta}(\phi(\cdot+x)) \\ &= \lim_{\delta\uparrow\infty}\lim_{D\uparrow\infty}\lim_{t\to\infty} v_{\delta,D}(t,m(t)+x) = \mathbb{E}\left[e^{-\tilde{C}(Z,C(\bar{\phi}))\tilde{Z}e^{-\sqrt{2}x}}\right]. \end{aligned} \tag{8.18}$$

To see that the constants $\tilde{C}(Z, C(\bar{\phi}))$ are strictly positive, one uses the Laplace functionals $\Psi_t(\phi)$ are bounded from above by

$$\mathbb{E}\left[\exp\left(-\phi\left(\max_{i\le n(bt)} x_i(bt) + \max_{j\le n_1((1-b)t)} x_j^1((1-b)t) - m(t)\right)\right)\right] \tag{8.19}$$

Here we used that the offspring of any of the particles at time bt has the same law. So the sum of the two maxima in the expression above has the same distribution as the largest descendent at time t off the largest particle at time bt. The limit of Eq. (8.19) as $t \uparrow \infty$ exists and is strictly smaller than 1 by the convergence in law of the recentered maximum of a standard BBM. But this implies the positivity of the constants $\tilde{C}$. Hence a limiting point process exists. Finally, one may easily check that the right hand side of (8.18) coincides with the Laplace functional of the point process defined in (1.15) by basically repeating the computations above. □

Remark 8.2. *Note that in particular, the structure of the variance profile is contained in the constant $\tilde{C}(D, Z, C(\bar{\phi}, \delta))$ and that also the information on the structure of the limiting point process is contained in this constant. In fact, we see that in all cases we have considered in this paper, the Laplace functional of the limiting process has the form*

$$\lim_{t\uparrow\infty} \Psi_t(\phi(\cdot + x)) = \mathbb{E}\exp\left(-C(\phi)Me^{-\sqrt{2}x}\right), \tag{8.20}$$

where M is a martingale limit (either Y of Z) and C is a map from the space of positive continuous functions with compact support to the real numbers. This function contains all the information on the specific limiting process. This is compatible with the finding in [16] in the case where the speed is a concave function of s/t. The universal form (8.20) is thus misleading and without knowledge of the specific form of $C(\phi)$, (8.20) contains almost no information.

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CHAPTER 3

Variable Speed Branching Brownian Motion 1. Extremal Processes in the Weak Correlation Regime

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Variable Speed Branching Brownian Motion 1. Extremal Processes in the Weak Correlation Regime

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Abstract. We prove the convergence of the extremal processes for variable speed branching Brownian motions where the "speed functions", that describe the time-inhomogeneous variance, lie strictly below their concave hull and satisfy a certain weak regularity condition. These limiting objects are universal in the sense that they only depend on the slope of the speed function at 0 and the final time t. The proof is based on previous results for two-speed BBM obtained in Bovier and Hartung (2014) and uses Gaussian comparison arguments to extend these to the general case.

1. Introduction

Gaussian processes indexed by trees is a topic that received a lot of attention, in particular in the context of spin glass theory (see e.g. Bovier (2006); Talagrand (2011a,b); Panchenko (2013)) through the so-called Generalised Random Energy Models (GREM), introduced and studied by Derrida (1985); Gardner and Derrida (1986a,b). Other contexts where such processes appeared are branching random walks (see e.g.Bramson (1978b); Shi (2011); Zeitouni (2013)) and branching Brownian motion (see e.g. Moyal (1957); McKean (1975); Bramson (1978a, 1983); Derrida and Spohn (1988)).

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One of the issues of interest in this context is to understand the structure of the extremal processes that arise in these models in the limit when the size of the tree tends to infinity. A Gaussian process on a tree is characterised fully by the tree and by its covariance, which in the models we are interested in is a function of the genealogical distance on the tree. In the classical models of branching random walk and branching Brownian motion, the covariance is a linear function of the tree-distance. In the context of the GREM, the tree is a binary tree with N levels; another popular tree is a supercritical Galton-Watson tree (see, e.g. Athreya and Ney (1972)). These models generalise branching Brownian motion and were first introduced, to our knowledge, in Derrida and Spohn (1988).

In this paper we focus on this latter class of models. They can be constructed as follows. On some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define a supercritical Galton-Watson (GW) tree. The offspring distribution, $\{p_k\}_{k\in\mathbb{N}}$, is normalised for convenience such that $\sum_{i=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$, and the second moment, $K = \sum_{k=1}^{\infty} k(k-1)p_k$ is assumed finite. We fix a time horizon $t > 0$. We denote the number of individuals ("leaves") of the tree at time t by $n(t)$ and label the leaves at time t by $i_1(t), i_2(t), \ldots, i_{n(t)}(t)$. For given t and for $s \leq t$, it is convenient to let $i_k(s)$ denote the ancestor of particle $i_k(t)$ at time s. Of course, in general there will be several indices k, ℓ such that $i_k(s) = i_\ell(s)$. The time of the most recent common ancestor of $i_k(t)$ and $i_\ell(s)$ is given, for $s, r \leq t$, by

$$d(i_k(r), i_\ell(s)) \equiv \sup\{u \leq s \wedge r : i_k(u) = i_\ell(u)\}. \tag{1.1}$$

We denote by $\mathcal{F}_s^{\mathrm{tree}}, s \in \mathbb{R}_+$ the σ-algebra generated by the Galton-Watson process up to time s. On the same probability space we will now construct, for given t, and for any realisation of the GW tree, a Gaussian process as follows.

Let $A : [0,1] \to [0,1]$ be a right-continuous non-decreasing function. We define a Gaussian process, x, labelled by the tree (up to time t), i.e. by $\{i_k(s)\}_{1\leq k\leq n(t)}^{0\leq s\leq t}$, with covariance, for $0 \leq s, r \leq t$ and $k, \ell \leq n(t)$

$$\mathbb{E}\left[x_k(s)x_\ell(r)\right] = tA\left(t^{-1}d(i_k(r), i_\ell(s))\right). \tag{1.2}$$

The existence of such a process is shown easily through a construction as time changed branching Brownian motion. Note first that, in the case when $A(x) = x$, this process is standard branching Brownian motion Moyal (1957); Skorohod (1964). For general A, the models can by constructed from *time changed* Brownian motion as follows. Let

$$\Sigma^2(s) = tA(s/t). \tag{1.3}$$

Note that Σ^2 is almost everywhere differentiable and denote by $\sigma^2(x)$ its derivative wherever it exists. Define the process $\{B_s^\Sigma\}_{0\leq s\leq t}$ on $[0,t]$ as time change of ordinary Brownian motion, B, via

$$B_s^\Sigma = B_{\Sigma^2(s)}. \tag{1.4}$$

Branching Brownian motion with speed function Σ^2 is constructed like ordinary Brownian motion, except that if a particle splits at some time $s < t$, then the offspring particles perform variable speed Brownian motions with speed function Σ^2, i.e. they are independent copies of $\{B_r^\Sigma - B_s^\Sigma\}_{t\geq r\geq s}$, all starting at the position of the parent particle at time s. We refer to these processes as *variable speed branching Brownian motion.* This class of processes, labelled by the different choices of functions A, provides an interesting set of examples to study the possible limiting extremal processes for correlated random variables. The ultimate goal will be to describe the extremal processes in dependence on the function A.

Remark 1.1. Strictly speaking, we are not talking about a single stochastic process, but about a family, $\{x_k^t(s), k \leq n(s)\}_{s \leq t}^{t \in \mathbb{R}_+}$, of processes with finite time horizon, indexed by that horizon, t. That dependence on t is usually not made explicit in order not to overburden the notation.

Branching Brownian motion has received a lot of attention over the last decades, with a strong focus on the properties of extremal particles. We mention the seminal contributions of McKean (1975); Bramson (1978a, 1983); Lalley and Sellke (1987), and Chauvin and Rouault (1988, 1990) on the connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation Fisher (1937); Kolmogorov et al. (1937) and on the distribution of the rescaled maximum. In recent years, there has been a revival of interest in BBM with numerous contributions, including the construction of the full extremal process by Aïdékon et al. (2013) and Arguin et al. (2013). For a review of these developments see, e.g., the recent survey by Gouéré (2014) or the lecture notes Bovier (2015). Variable speed branching Brownian motion (as well as random walk) has recently been investigated by Fang and Zeitouni (2012a,b); Maillard and Zeitouni (2013); Mallein (2013), and the present authors Bovier and Hartung (2014).

Naturally, the same construction can be done for any other family of trees. It is widely believed (see Zeitouni (2013)) that the resulting structures are very similar, with only details depending on the underlying tree model. More importantly, it is believed that the extremal structure in more general Gaussian processes, such as mean field spin glasses Bolthausen and Kistler (2006, 2009) or the Gaussian free field Zeitouni (2013) are of the same type; considerable progress in this direction has been made recently by Bramson et al. (2013) and by Biskup and Louidor (2013).

We are interested in understanding the nature of the extremes of our processes in dependence on the properties of the covariance functions A. The case when A is a step function with finitely many steps corresponds to Derrida's GREMs Gardner and Derrida (1986b); Bovier and Kurkova (2004a), the only difference being that the deterministic binary tree of the GREM is replaced by a Galton-Watson tree. It is very easy to treat this case.

The case when A is arbitrary has been dubbed CREM in Bovier and Kurkova (2004b) (and treated for binary regular trees). In that case the leading order of the maximum was obtained, as well as the genealogical description of the Gibbs measures; this analysis carries over mutando mutandis to the analogous BBM situation. The finer analysis of the extremes is, however, much more subtle and in general still open. Fang and Zeitouni (2012b) have obtained the order of the corrections (namely $t^{1/3}$) in the case when A is strictly concave and continuous. These corrections come naturally from the probability of a Brownian bridge to stay away from a curved line, which was earlier analysed in Ferrari and Spohn (2005). There are, however, no results on the extremal process or the law of the maximum.

Another rather tractable situation occurs when A is a piecewise linear function. The simplest case here corresponds to choosing a speed that takes just two values, i.e.

$$\sigma^2(s) = \begin{cases} \sigma_1^2, & \text{for } 0 \leq s < tb, \\ \sigma_2^2, & \text{for } bt \leq s \leq t, \end{cases} \tag{1.5}$$

with $\sigma_1^2 b + \sigma_2^2(1-b) = 1$. In this case, Fang and Zeitouni (2012b) have obtained the correct order of the logarithmic corrections. This case was fully analysed in a recent paper of ours Bovier and Hartung (2014), where we provide the construction of the extremal processes.

In the present paper, we present the full picture in the case where $A(x) < x$ for all $x \in (0,1)$, and the slopes of A at 0 and at 1 are different from 1. We show that there is a large degree of universality in that the limiting extremal processes are those that emerged in the two-speed case, and that they depend only on the slopes of A at 0 and at 1.

The critical cases, $A(x) \leq x$, involve, besides the well-understood standard BBM, a number of different situations that can be quite tricky, and we postpone this analysis to a forthcoming publication.

1.1. *Results.* We need some mild technical assumptions on the covariance function. Let $A : [0,1] \to [0,1]$ be a right-continuous, non-decreasing function that satisfies the following three conditions:

(A1) For all $x \in (0,1)$: $A(x) < x$, $A(0) = 0$ and $A(1) = 1$.

(A2) There exists $\delta_b > 0$ and functions $\overline{B}(x)$, $\underline{B}(x) : [0,1] \to [0,1]$ that are twice differentiable in $[0, \delta_b]$ with bounded second derivatives, such that

$$\underline{B}(x) \leq A(x) \leq \overline{B}(x), \quad \forall x \in [0, \delta_b] \tag{1.6}$$

with $\overline{B}'(0) = \underline{B}'(0) \equiv A'(0)$.

(A3) There exists $\delta_e > 0$ and functions $\overline{C}(x)$, $\underline{C}(x) : [0,1] \to [0,1]$ that are twice differentiable in $[1-\delta_e, 1]$ with bounded second derivatives, such that

$$\underline{C}(x) \leq A(x) \leq \overline{C}(x), \quad \forall x \in [1-\delta_e, 1] \tag{1.7}$$

with $\overline{C}'(1) = \underline{C}'(1) \equiv A'(1)$. The case $A'(1) = +\infty$ is allowed. This is to be understood in the sense that, for all $\rho < \infty$, there exists $\varepsilon > 0$ such that, for all $x \in [1-\varepsilon, 1]$, $A(x) \leq 1 - \rho(1-x)$.

For standard BBM, $\bar{x}(t)$, recall that Bramson (1978a) and Lalley and Sellke (1987) have shown that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k\leq n(t)} \bar{x}_k(t) - m(t) \leq y \right) = \omega(x) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \tag{1.8}$$

where $m(t) \equiv \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, Z is a random variable, the limit of the so called *derivative martingale*, and C is a constant.

In Arguin et al. (2013) (see also Aïdékon et al. (2013) for a different proof) it was shown that the extremal process,

$$\lim_{t\uparrow\infty} \widetilde{\mathcal{E}}_t \equiv \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\bar{x}_k(t)-m(t)} = \widetilde{\mathcal{E}}, \tag{1.9}$$

exists in law, and $\widetilde{\mathcal{E}}$ is of the form

$$\widetilde{\mathcal{E}} = \sum_{k,j} \delta_{\eta_k + \Delta_j^{(k)}}, \tag{1.10}$$

where η_k is the k-th atom of a Cox process Cox (1955)) directed by the random measure $CZ\mathrm{e}^{-\sqrt{2}y}dy$, with C and Z as before. $\Delta_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, which are the limits in law of

$$\sum_{j\leq n(t)} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)}\tilde{x}_j(t)}, \tag{1.11}$$

where $\tilde{x}(t)$ is BBM conditioned on the event $\max_{j\leq n(t)}\tilde{x}_j(t)\geq\sqrt{2}t$.

The main result of the present paper is the following theorem.

Theorem 1.2. *Assume that $A:[0,1]\to[0,1]$ satisfies (A1)-(A3). Let $A'(0)=\sigma_b^2<1$ and $A'(1)=\sigma_e^2>1$. Let $\tilde{m}(t)=\sqrt{2}t-\frac{1}{2\sqrt{2}}\log t$. Then there is a constant $\widetilde{C}(\sigma_e)$ depending only on σ_e and a random variable Y_{σ_b} depending only on σ_b such that*

(i)

$$\lim_{t\uparrow\infty}\mathbb{P}\left(\max_{1\leq i\leq n(t)} x_i(t)-\tilde{m}(t)\leq x\right)=\mathbb{E}\left[\mathrm{e}^{-\widetilde{C}(\sigma_e)Y_{\sigma_b}\mathrm{e}^{-\sqrt{2}x}}\right]. \tag{1.12}$$

(ii) *The point process*

$$\sum_{k\leq n(t)}\delta_{x_k(t)-\tilde{m}(t)}\to\mathcal{E}_{\sigma_b,\sigma_e}=\sum_{i,j}\delta_{p_i+\sigma_e\Lambda_j^{(i)}}, \tag{1.13}$$

as $t\uparrow\infty$, in law, where the p_i are the atoms of a Cox process on $\mathbb{R}$ directed by the random measure $\widetilde{C}(\sigma_e)Y_{\sigma_b}\mathrm{e}^{-\sqrt{2}x}dx$, and the $\Lambda^{(i)}$ are the limits of the processes as in (1.11), but conditioned on the event $\{\max_k\tilde{x}_k(t)\geq\sqrt{2}\sigma_e t\}$.

(iii) *If $A'(1)=\infty$, then $\widetilde{C}(\infty)=1/\sqrt{4\pi}$, and $\Lambda^{(i)}=\delta_0$, i.e. the limiting process is a Cox process.*

The random variable Y_{σ_b} is the limit of the uniformly integrable martingale

$$Y_{\sigma_b}(s)=\sum_{i=1}^{n(s)}\mathrm{e}^{-s(1+\sigma_b^2)+\sqrt{2}\sigma_b\bar{x}_i(s)}, \tag{1.14}$$

where $\bar{x}_i(s)$ is standard branching Brownian motion.

Remark 1.3. In Theorem 7.7 of Bovier and Hartung (2014) the constant $\widetilde{C}(\sigma_e)$ is characterised by the tail behaviour of solutions to the F-KPP equation, namely

$$\widetilde{C}(\sigma_e)\equiv\sigma_e\lim_{t\to\infty}e^{\sqrt{2}x}e^{x^2/2t}t^{1/2}u(t,x+\sqrt{2}t), \tag{1.15}$$

where $x\equiv\sqrt{2}(\sigma_e-1)t$, and u solves the F-KPP equation

$$\partial_t u(t,x)=\frac{1}{2}\partial_x^2u(t,x)+(1-u(t,x))-\sum_{k=1}^{\infty}p_k(1-u(t,x))^k, \tag{1.16}$$

with initial condition $u(0,x)=\mathbb{1}_{x\leq 0}$.

Remark 1.4. The special case of Theorem 1.2 when A consists of two linear segments was obtained in Bovier and Hartung (2014). Theorem 1.2 shows that the limiting objects under conditions $(A1)-(A3)$ are universal and depend only on the slopes of the covariance function A at 0 and at 1. This could have been guessed, but the rigorous proof turns out to be quite involved. Note that $\sigma_e=\infty$ is allowed. In that case the extremal process is just a mixture of Poisson point processes. If $\sigma_b=0$, then Y_{σ_b} is just an exponential random variable of mean 1. We call $(Y_{\sigma_b}(s))_{s\in\mathbb{R}_+}$ the *McKean martingale.*

1.2. *Outline of the proof.* The proof of Theorem 1.2 is based on the corresponding result obtained in Bovier and Hartung (2014) for the case of two speeds, and on a Gaussian comparison method. We start by showing the localisation of paths, namely that the paths of all particles that reach a hight of order $\tilde{m}(t)$ at time t has to lie within a certain tube. Next, we show tightness of the extremal process.

The remainder of the paper is then concerned with proving the convergence of the finite dimensional distributions through Laplace transforms. We introduce auxiliary two speed BBM's whose covariance functions approximate A well around 0 and 1. Moreover we choose them in such a way that their covariance functions lie above respectively below A in a neighbourhood of 0 and 1 (see Figure 1.1).

We then use Gaussian comparison methods to compare the Laplace transforms. The Gaussian comparison comes in three main steps. In a first step we introduce the usual interpolating process and introduce a localisation condition on its paths. In a second step we justify a certain integration by parts formula, that is adapted to our setting. Finally, the resulting quantities are decomposed into a part with controlled sign and a part that converges to zero.

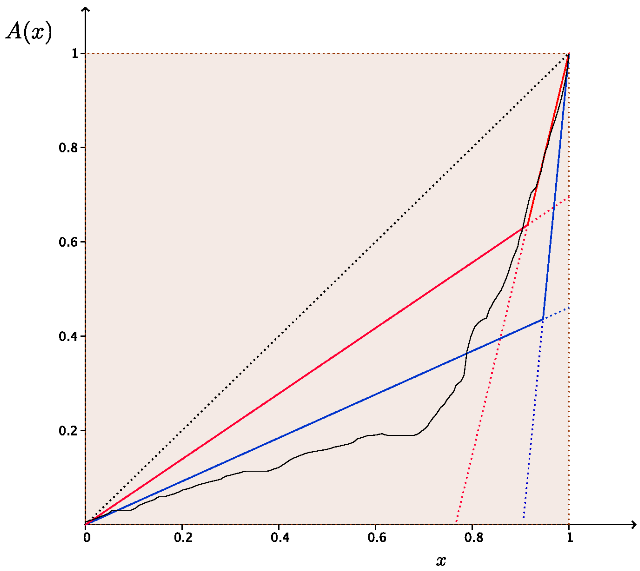


FIGURE 1.1. Gaussian Comparison: The extremal process of BBM with covariance A (black curve) is compared to process with covariances functions $\overline{A}$ (red curve), respectively $\underline{A}$ (blue, curve).

2. Localization of paths

In this section we show where the paths of particles that are extreme at time t are localised. This is essentially inherited from properties of the standard Brownian bridge. For a given speed function Σ^2, and a subinterval $I \subset [0,t]$, define the following events on the space of paths, $X : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathcal{T}^{\gamma}_{t,I,\Sigma^2} = \left\{ X \middle| \forall s : s \in I : \left| X(s) - \tfrac{\Sigma^2(s)}{t} X(t) \right| < (\Sigma^2(s) \wedge (t - \Sigma^2(s)))^{\gamma} \right\}. \tag{2.1}$$

Proposition 2.1. *Let x denote variable speed BBM with covariance function A. For $0 \le r < t$, set $I_r \equiv \{s : \Sigma^2(s) \in [r, t-r]\}$. For any $\gamma > \frac{1}{2}$ and for all $d \in \mathbb{R}$, for all $\epsilon > 0$, there exists $r_0 < \infty$ such that, for $r > r_0$ and for all $t > 3r$,*

$$\mathbb{P}\left(\exists k \le n(t) : \{x_k(t) > \tilde{m}(t) + d\} \wedge \left\{ x_k \notin \mathcal{T}^{\gamma}_{t,I_r,\Sigma^2} \right\} \right) < \epsilon. \tag{2.2}$$

To prove Proposition 2.1 we need the following lemma on Brownian bridges (see Bramson (1983)).

Lemma 2.2. *Let $\gamma > \frac{1}{2}$. Let ξ be a Brownian bridge from 0 to 0 in time t. Then, for all $\epsilon > 0$, there exists $r_0 < \infty$ such that, for $r > r_0$ and for all $t > 3r$,*

$$\mathbb{P}\left(\exists s \in [r, t-r] : |\xi(s)| > (s \wedge (t-s))^{\gamma}\right) < \epsilon. \tag{2.3}$$

More precisely,

$$\mathbb{P}\left(\exists s \in [r, t-r] : |\xi(s)| > (s \wedge (t-s))^{\gamma}\right) < 8 \sum_{k=\lfloor r \rfloor}^{\infty} k^{\frac{1}{2}-\gamma} \mathrm{e}^{-k^{2\gamma-1}/2}. \tag{2.4}$$

Proof: The probability in (2.3) is bounded from above by

$$\begin{aligned} & \sum_{k=\lceil r \rceil}^{\lceil t-r \rceil} \mathbb{P}\left(\exists s \in [k-1, k] : |\xi(s)| > (s \wedge (t-s))^{\gamma}\right) \\ \le \quad & 2 \sum_{k=\lceil r \rceil}^{\lceil t/2 \rceil} \mathbb{P}\left(\exists s \in [k-1, k] : |\xi(s)| > (s \wedge (t-s))^{\gamma}\right), \end{aligned} \tag{2.5}$$

by the reflection principle for the Brownian bridge. This is bounded from above by

$$2 \sum_{k=\lceil r \rceil}^{\lceil t/2 \rceil} \mathbb{P}\left(\exists s \in [0, k] : |\xi(s)| > (k-1)^{\gamma}\right). \tag{2.6}$$

Using the bound of Lemma 2.2 (b) of Bramson (1983) we have

$$P\left(\exists s \in [0, k] : |\xi(s)| > (k-1)^{\gamma}\right) \le 4(k-1)^{\frac{1}{2}-\gamma} \mathrm{e}^{-(k-1)^{2\gamma-1}/2}. \tag{2.7}$$

Using this bound for each summand in (2.6) we obtain (2.4). Since the sum on the right-hand side of (2.4) is finite (2.3) follows. □

Proof of Proposition 2.1: Using a first moment method, the probability in (2.2) is bounded from above by

$$\mathrm{e}^t \mathbb{P}\left(B_{\Sigma^2(t)} > \tilde{m}(t) + d, B_{\Sigma^2(\cdot)} \notin \mathcal{T}^{\gamma}_{t,I_r,\Sigma^2} \right). \tag{2.8}$$

Since $\Sigma^2(s)$ is an non-decreasing function on $[0,t]$ with $\Sigma^2(t) = t$, the expression in (2.8) is bounded from above by

$$\mathrm{e}^t \mathbb{P}\left(\{B_t > \tilde{m}(t) + d\} \wedge \left\{ \exists s \in [r, t-r] : \left| B_s - \frac{s}{t} B_t \right| > (s \wedge (t-s))^{\gamma} \right\} \right). \tag{2.9}$$

Now, $\xi(s) \equiv B_s - \frac{s}{t}B_t$ is the Brownian bridge from 0 to 0 in time t, and it is well known (see e.g. Lemma 2.1 in Bramson (1983)) that $\xi(s)$ is independent of B_t, for all $s \in [0,t]$. Therefore, (2.9) is equal to

$$\mathrm{e}^t \mathbb{P}\left(B_t > \tilde{m}(t) + d\right) \mathbb{P}\left(\exists s \in [r, t-r] : |\xi(s)| > (s \wedge (t-s))^\gamma\right). \tag{2.10}$$

Using the standard Gaussian tail bound,

$$\int_u^\infty \mathrm{e}^{-x^2/2} dx \leq u^{-1} \mathrm{e}^{-u^2/2}, \quad \text{for } u > 0, \tag{2.11}$$

we have

$$\begin{aligned} \mathrm{e}^t \mathbb{P}\left(B_t > \tilde{m}(t) + d\right) &\leq \mathrm{e}^t \frac{\sqrt{t}}{\sqrt{2\pi}(\tilde{m}(t)+d)} \mathrm{e}^{-(\tilde{m}(t)+d)^2/2t} \\ &\leq \frac{t}{\sqrt{2\pi}(\tilde{m}(t)+d)} \mathrm{e}^{-\sqrt{2}d} \leq M, \end{aligned} \tag{2.12}$$

for some constant M (depending on d), if t is large enough. By Lemma 2.2 we can find r_0 large enough such that for all $r \geq r_0$ and $t > 3r$,

$$\mathbb{P}\left(\exists s \in [r, t-r] : |\xi(s)| > (s \wedge (t-s))^\gamma\right) < \epsilon / M. \tag{2.13}$$

The bounds (2.12) and (2.13) imply that (2.10) is smaller than ϵ. □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2 assuming Proposition 3.2 below, whose proof will be postponed to the following two sections.

Proof of Theorem 1.2: We show the convergence of the extremal process

$$\mathcal{E}_t = \sum_{k \leq n(t)} \delta_{x_k(t) - \tilde{m}(t)} \tag{3.1}$$

by showing the convergence of the finite dimensional distributions and tightness. Tightness of $(\mathcal{E}_t)_{t \geq 0}$ is implied by the following bound on the number of particles above a level d (see Resnick (1987), Lemma 3.20).

Proposition 3.1. *For any $d \in \mathbb{R}$ and $\epsilon > 0$, there exists $N = N(\epsilon, d)$ such that, for all t large enough,*

$$\mathbb{P}\left(\mathcal{E}_t[d, \infty) \geq N\right) < \epsilon. \tag{3.2}$$

Proof: By a first order Chebyshev inequality, for all t large enough,

$$\mathbb{P}\left(\mathcal{E}_t[d, \infty) \geq N\right) \leq \frac{1}{N} \mathrm{e}^t \mathbb{P}\left(B_t > \tilde{m}(t) + d\right) \leq \frac{M}{N} \tag{3.3}$$

by (2.12), where $M > 0$ is a constant that depends on d. Choosing $N > M/\epsilon$ yields Proposition 3.1. □

To show the convergence of the finite dimensional distributions define, for $u \in \mathbb{R}$,

$$\mathcal{N}_u(t) = \sum_{i=1}^{n(t)} \mathbb{1}_{x_i(t) - \tilde{m}(t) > u}, \tag{3.4}$$

that counts the number of points that lie above u. Moreover, we define the corresponding quantity for the process $\mathcal{E}_{\sigma_b,\sigma_e}$ (defined in (1.13)),

$$\mathcal{N}_u = \sum_{i,j} \mathbb{1}_{p_i+\sigma_e\Lambda_j^{(i)}>u}. \tag{3.5}$$

Observe that, in particular,

$$\mathbb{P}\left(\max_{1\le i\le n(t)} x_i(t) - \tilde{m}(t) \le u\right) = \mathbb{P}\left(\mathcal{N}_u(t) = 0\right). \tag{3.6}$$

The key step in the proof of Theorem 1.2 is the following proposition, that asserts the convergence of the finite dimensional distributions of the process $\mathcal{E}_t$.

Proposition 3.2. *For all $k \in \mathbb{N}$ and $u_1, \dots, u_k \in \mathbb{R}$*

$$\{\mathcal{N}_{u_1}(t), \dots, \mathcal{N}_{u_k}(t)\} \xrightarrow{d} \{\mathcal{N}_{u_1}, \dots, \mathcal{N}_{u_k}\} \tag{3.7}$$

as $t \uparrow \infty$.

The proof of this proposition will be postponed to the following sections.

Assuming the proposition, we can now conclude the proof of the theorem. The distribution of $\{\mathcal{N}_{u_1}(t), \dots, \mathcal{N}_{u_k}(t)\}$ for all $k \in \mathbb{N}, u_1, \dots, u_k \in \mathbb{R}$ characterise the finite dimensional distributions of the point process $\mathcal{E}_t$ since the class of sets $\{(u,\infty), u \in \mathbb{R}\}$ form a Π-system that generates $\mathcal{B}(\mathbb{R})$. Hence (3.7) implies the convergence of the finite dimensional distributions of $\mathcal{E}_t$ (see, e.g., Proposition 3.4 in Resnick (1987)).

Combining this observation with Propositions 3.1, we obtain Assertion (ii) of Theorem 1.2. Assertion (i) follows immediately from Eq. (3.6).

To prove Assertion (iii), we need to show that, as $\sigma_e^2 \uparrow \infty$, it holds that $\widetilde{C}(\sigma_e) \uparrow 1/\sqrt{4\pi}$ and the processes $\Lambda^{(i)}$ converge to the trivial process δ_0. Then,

$$\mathcal{E}_{\sigma_b,\infty} = \sum_i \delta_{p_i}, \tag{3.8}$$

where $(p_i, i \in \mathbb{N})$ are the points of a Cox process directed by the random measure $\frac{1}{\sqrt{4\pi}} Y_{\sigma_b} \mathrm{e}^{-\sqrt{2}x} dx$.

Lemma 3.3. *The point process $\mathcal{E}_{\sigma_b,\sigma_e}$ converges in law, as $\sigma_e \uparrow \infty$, to the point process $\mathcal{E}_{\sigma_b,\infty}$.*

Proof: The proof of Lemma 3.3 is based on a result concerning the cluster processes $\Lambda^{(i)}$. We write Λ_{σ_e} for a single copy of these processes and add the subscript to make the dependence on the parameter σ_e explicit. We recall from Bovier and Hartung (2014) that the process Λ_{σ_e} is constructed as follows. Define the processes $\overline{\mathcal{E}}_{\sigma_e}$ as the limits of the point processes

$$\overline{\mathcal{E}}_{\sigma_e}^t \equiv \sum_{k=1}^{n(t)} \delta_{x_k(t)-\sqrt{2}\sigma_e t}, \tag{3.9}$$

where x is standard BBM at time t conditioned on the event $\{\max_{k\le n(t)} x_k(t) > \sqrt{2}\sigma_e t\}$. We show here that, as σ_e tends to infinity, the processes $\overline{\mathcal{E}}_{\sigma_e}$ converge to a point process consisting of a single atom at 0. More precisely, we show that

$$\lim_{\sigma_e\uparrow\infty}\lim_{t\uparrow\infty} \mathbb{P}\left(\overline{\mathcal{E}}_{\sigma_e}^t([-R,\infty)) > 1 \big| \max_{k\le n(t)} x_k(t) > \sqrt{2}\sigma_e t\right) = 0. \tag{3.10}$$

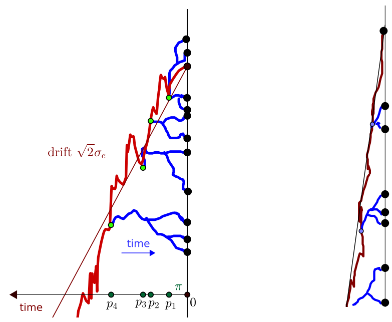


FIGURE 3.2. The cluster process seen from infinity for σ_e small (left) and σ_e very large (right)

Now,

$$\begin{aligned}
&\mathbb{P}\left(\overline{\mathcal{E}}^t_{\sigma_e}([-R,\infty)) > 1 \big| \max_{k\leq n(t)} x_k(t) > \sqrt{2}\sigma_e t\right) \\
&\leq \mathbb{P}\left(\operatorname{supp} \overline{\mathcal{E}}^t_{\sigma_e} \cap [0,\infty) \neq \emptyset \wedge \overline{\mathcal{E}}^t_{\sigma_e}([-R,\infty)) > 1 \big| \max_{k\leq n(t)} x_k(t) > \sqrt{2}\sigma_e t\right) \\
&\leq \int_0^\infty \mathbb{P}\left(\operatorname{supp} \overline{\mathcal{E}}^t_{\sigma_e} \cap dy \neq \emptyset \wedge \overline{\mathcal{E}}^t_{\sigma_e}([-R,\infty)) > 1 \big| \max_{k\leq n(t)} x_k(t) > \sqrt{2}\sigma_e t\right) \\
&= \int_0^\infty \mathbb{P}\left(\operatorname{supp} \overline{\mathcal{E}}^t_{\sigma_e} \cap dy \neq \emptyset \big| \max_{k\leq n(t)} x_k(t) > \sqrt{2}\sigma_e t\right) \\
&\qquad \times \mathbb{P}\left(\overline{\mathcal{E}}^t_{\sigma_e}([-R,\infty)) > 1 \big| \operatorname{supp} \overline{\mathcal{E}}^t_{\sigma_e} \cap dy \neq \emptyset\right). \qquad (3.11)
\end{aligned}$$

But $\mathbb{P}\left(\cdot \big| \operatorname{supp} \overline{\mathcal{E}}^t_{\sigma_e} \cap dy \neq \emptyset\right) \equiv P_{t,y+\sqrt{2}\sigma_e}(\cdot)$ is the *Palm measure* on BBM, i.e. the conditional law of BBM given that there is a particle at time t in dy (see Kallenberg (1986, Theorem 12.8)) Chauvin et al. (1991, Theorem 2) describe the tree under the Palm measure $P_{t,z}$ as follows. Pick one particle at time t at the location z. Then pick a *spine*, Y, which is a Brownian bridge from 0 to z in time t. Next pick a Poisson point process π on $[0,t]$ with intensity 2. For each point $p \in \pi$ start a random number ν_p of independent branching Brownian motions $(\mathcal{B}^{Y(p),i}, i \leq \nu_p)$ starting at $Y(p)$. The law of ν is given by the size biased distribution, $\mathbb{P}(\nu_p = k-1) \sim \frac{kp_k}{2}$. See Figure 3.2. Now let $z = \sqrt{2}\sigma_e t + y$ for $y \geq 0$. Under the Palm measure, the point process $\overline{\mathcal{E}}_{\sigma_e}(t)$ then takes the form

$$\overline{\mathcal{E}}_{\sigma_e}(t) \overset{\text{law}}{=} \delta_y + \sum_{p\in\pi, i<\nu_p} \sum_{j=1}^{n_{Y(p),i}(p)} \delta_{\mathcal{B}_j^{Y(p),i}(t-p) - \sqrt{2}\sigma_e t}. \qquad (3.12)$$

Since, for $1 > \gamma > 1/2$,

$$\lim_{\sigma_e\uparrow\infty}\lim_{t\uparrow\infty}\mathbb{P}\left(\forall s\geq\sigma_e^{-1/2}:Y(t-s)-y+\sqrt{2}\sigma_e s\in[-(\sigma_e s)^\gamma,(\sigma_e s)^\gamma]\right)=1, \quad (3.13)$$

if we define the set

$$\mathcal{G}^t_{\sigma_e}\equiv\left\{Y:\forall t\geq s\geq\sigma_e^{-1/2}:Y(t-s)-y+\sqrt{2}\sigma_e s\in[-(\sigma_e s)^\gamma,(\sigma_e s)^\gamma]\right\}, \quad (3.14)$$

it will suffice to show that, for all $R\in\mathbb{R}_+$,

$$\lim_{\sigma_e\uparrow\infty}\lim_{t\uparrow\infty}\mathbb{P}\left(\exists p\in\pi,i<\nu_p,j:\mathcal{B}_j^{Y(p),i}(t-p)\geq y-R\wedge Y\in\mathcal{G}^t_{\sigma_e}\right)=0. \quad (3.15)$$

The probability in (3.15) is bounded by

$$\begin{aligned}
&\mathbb{P}\left(\exists p\in\pi,i\leq\nu_p,j:\mathcal{B}_j^{Y(p),i}(t-p)\geq y-R)\wedge Y\in\mathcal{G}^t_{\sigma_e}\right)\\
\leq\ &\mathbb{E}\left[\int_0^t\sum_{i=1}^{\nu_p}\mathbb{1}_{\mathcal{B}_j^{Y(p),i}(t-p)>y-R}\mathbb{1}_{Y\in\mathcal{G}_{\sigma_e}}\pi(dp)\right]\\
\leq\ &\mathbb{E}\left[\int_0^t\mathbb{E}\left[\sum_{i=1}^{\nu_p}\mathbb{1}_{\max_j\mathcal{B}_j^{Y(p),i}(t-p)\geq y-R}\mathbb{1}_{Y\in\mathcal{G}^t_{\sigma_e}}\middle|\mathcal{F}^\pi\right]\pi(dp)\right]\\
\leq\ &\int_0^t 2K\mathbb{P}\left(\max_j\mathcal{B}_j^{Y(t-s)}\geq y-R\wedge Y\in\mathcal{G}^t_{\sigma_e}\right)ds. \quad (3.16)
\end{aligned}$$

Here we used the independence of the offspring BBM and that the conditional probability given the σ-algebra $\mathcal{F}^\pi$ generated by the Poisson process π appearing in the integral over π depends only on p. For the integral over s up to $1/\sigma_\epsilon^{1/2}$, we just bound the integrand by $2K$. For larger values, we use the localisation provided by the condition that $Y\in\mathcal{G}_{\sigma_e}$, to get that the right hand side of (3.16) is not larger than

$$2K\int_0^{\sigma_e^{-1/2}}ds+2K\int_{\sigma_e^{-1/2}}^t e^s\mathbb{P}(B(s)>-R+\sqrt{2}\sigma_e s-(\sigma_e s)^\gamma)ds. \quad (3.17)$$

(3.17) is by (2.11) bounded from above by

$$2K\sigma_e^{-1/2}+2K\int_{\sigma_e^{-1/2}}^\infty e^{(1-\sigma_e^2)s+\sqrt{2}\sigma_e(R+(\sigma_e s)^\gamma)}ds. \quad (3.18)$$

From this it follows that (3.18) (which does no longer depend on t) converges to zero, as $\sigma_e\uparrow\infty$, for any $R\in\mathbb{R}$. Hence we see that

$$\mathbb{P}\left(\overline{\mathcal{E}}^t_{\sigma_e}([-R,\infty))>1\middle|\operatorname{supp}\overline{\mathcal{E}}^t_{\sigma_e}\cap dy\neq\emptyset\right)\downarrow 0, \quad (3.19)$$

uniformly in $y\geq 0$, as t and then σ_e tend to infinity. Next,

$$\begin{aligned}
&\int_0^\infty\mathbb{P}\left(\operatorname{supp}\overline{\mathcal{E}}^t_{\sigma_e}\cap dy\neq\emptyset\middle|\max_{k\leq n(t)}x_k(t)>\sqrt{2}\sigma_e t\right)\\
\leq&\int_0^\infty\mathbb{P}\left(\max_{k\leq n(t)}x_k(t)\geq\sqrt{2}\sigma_e t+y\middle|\max_{k\leq n(t)}x_k(t)>\sqrt{2}\sigma_e t\right). \quad (3.20)
\end{aligned}$$

But by Proposition 7.5 in Bovier and Hartung (2014) the probability in the integrand converges to $\exp(-\sqrt{2}\sigma_e y)$, as $t\uparrow\infty$. It follows from the proof that this convergence is unifomr in y, and hence by dominated convergence, the right-hand side of (3.20) is finite. Therefore, (3.10) holds. As a consequence, Λ_{σ_e} converges to δ_0.

It remains to show that the intensity of the Poisson process converges as claimed. Theorems 1 and 2 of Chauvin and Rouault (1988) relate the constant $\widetilde{C}(\sigma_e)$ defined by (1.15) to the intensity of the shifted BBM conditioned to exceed the level $\sqrt{2}\sigma_e t$ as follows:

$$\begin{aligned}
\frac{1}{\sqrt{4\pi}\widetilde{C}(\sigma_e)} &= \lim_{s\uparrow\infty} \frac{\mathbb{E}\left[\sum_k \mathbb{1}_{\bar{x}_k(s)>\sqrt{2}\sigma_e s}\right]}{\mathbb{P}\left(\max_k \bar{x}_k(s) > \sqrt{2}\sigma_e s\right)} \\
&= \lim_{s\uparrow\infty} \mathbb{E}\left[\sum_k \mathbb{1}_{\bar{x}_k(s)-\max_i \bar{x}_i(s)>\sqrt{2}\sigma_e s-\max_i \bar{x}_i(s)} \big| \max_k \bar{x}_k(s) > \sqrt{2}\sigma_e s\right] \\
&= \Lambda_{\sigma_e}((-E,0]),
\end{aligned} \tag{3.21}$$

where, by Theorem 7.5 in Bovier and Hartung (2014), E is a exponentially distributed random variable with parameter $\sqrt{2}\sigma_e$, independent of Λ_{σ_e}. As we have just shown that $\Lambda_{\sigma_e} \to \delta_0$, it follows that the right-hand side tends to one, as $\sigma_e \uparrow \infty$, and hence $\widetilde{C}(\sigma_e) \uparrow 1/\sqrt{4\pi}$. Hence the intensity measure of the PPP appearing in $\mathcal{E}_{\sigma_b,\sigma_e}$ converges to the desired intensity measure $\frac{1}{\sqrt{4\pi}} Y_{\sigma_b} \mathrm{e}^{-\sqrt{2}x} dx$. □

This proves Assertion (iii) of Theorem 1.2. □

4. Proof of Proposition 3.2

We prove Proposition 3.2 via convergence of Laplace transforms. For $u_1, \ldots, u_k \in \mathbb{R}, k \in \mathbb{N}$, define the Laplace transform of $\{\mathcal{N}_{u_1}(t), \ldots, \mathcal{N}_{u_k}(t)\}$,

$$\mathcal{L}_{u_1,\ldots,u_k}(t,c) = \mathbb{E}\left(\exp\left(-\sum_{l=1}^{k} c_l \mathcal{N}_{u_l}(t)\right)\right), \quad c = (c_1, \ldots, c_k)^t \in \mathbb{R}_+^k, \tag{4.1}$$

and analogously the Laplace transform $\mathcal{L}_{u_1,\ldots,u_k}(c)$ of $\{\mathcal{N}_{u_1}, \ldots, \mathcal{N}_{u_k}\}$. Proposition 3.2 is then a consequence of the next proposition.

Proposition 4.1. *For any $k \in \mathbb{N}$, $u_1, \ldots, u_k \in \mathbb{R}$ and $c_1, \ldots, c_k \in \mathbb{R}_+$*

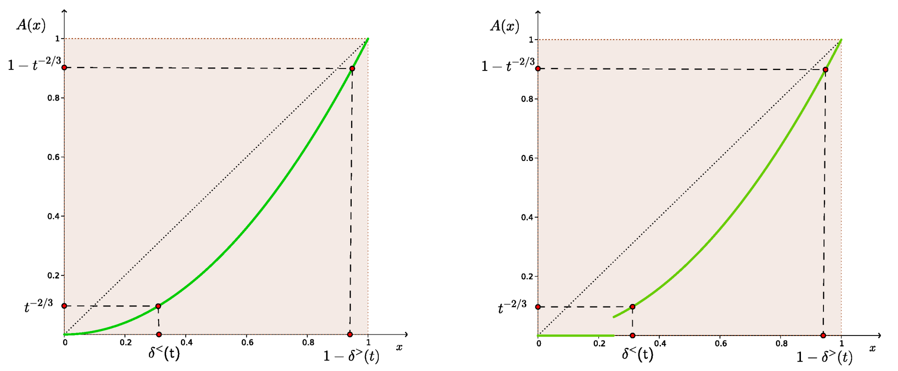
$$\lim_{t\to\infty} \mathcal{L}_{u_1,\ldots,u_k}(t,c) = \mathcal{L}_{u_1,\ldots,u_k}(c). \tag{4.2}$$

The proof of Proposition 4.1 comes in two main steps. First, we prove the result for the case of two speed BBM. This was done in our previous paper Bovier and Hartung (2014). In fact, we will need a slight extension of that result where we allow a slight dependence of the speeds on t. This will be given in the next subsection.

The second step is to show that, under the hypotheses of Theorem 1.2, the Laplace transforms can be well approximated by those of two speed BBM. This uses the classical Gaussian comparison argument in a slightly subtle way.

4.1. *Approximating two speed BBM. The case $A'(1) < \infty$.* It turns out that it is enough to compare the process with covariance function A with processes whose covariance function is piecewise linear with a single change in slope. We derive asymptotic upper and lower bounds by choosing these in such a way that the covariances near zero and near one are below, respectively above, that of the original process. We define

$$\begin{aligned}\delta^{<}(t) &= \sup\{x \in [0,1] : A(x) \leq t^{-2/3}\} \\ \delta^{>}(t) &= 1 - \inf\{x \in [0,1] : A(x) \geq 1 - t^{-2/3}\} \end{aligned} \tag{4.3}$$



(a) Case 1) $\sigma_b = 0$ but $\lim_{t\uparrow\infty} \delta^{<}(t) = 0$ (b) Case 2) $\sigma_b = 0$ but $\lim_{t\uparrow\infty} \delta^{<}(t) = \delta^{<} \neq 0$

FIGURE 4.3. Different cases for $\delta^{<}(t)$ and $\delta^{>}(t)$.

By Assumption (A1) it follows that $\lim_{t\uparrow\infty} \delta^{>}(t) = 0$.

Remark 4.2. If $\lim_{t\uparrow\infty} \delta^{<}(t) = \delta^{<} \neq 0$, then it follows from the definition of $\delta^{<}(t)$ that $A(x) = 0$ on $[0, \delta^{<}]$.

In the following formulas, we choose a parameter $n \in \mathbb{N}_{\geq 2}$ as follows. If in Assumption (A2) the functions $\overline{B}, \underline{B}$ can be chosen such that there exists $m \geq 2$, such that $\underline{B}^{(k)}(0) = \overline{B}^{(k)}(0) = 0$, for all $1 \leq k < m$, and in some finite interval $[0, \delta_b]$, both $|\underline{B}^{(m)}(x)|$ and $|\overline{B}^{(m)}(x)|$ are bounded by some constants $\underline{K}_1$, respectively $\overline{K}_1$, then we choose n as the largest of these integers. Otherwise, we choose $n = 2$. Moreover, let $|\overline{C}''(x)| \leq \overline{K}_2$ and $|\underline{C}''(x)| \leq \underline{K}_2$ for all $x \in [1 - \delta_e, 1]$. We define

$$\overline{\Sigma}^2(s) = t\overline{A}(s/t) \tag{4.4}$$

and

$$\underline{\Sigma}^2(s) = t\underline{A}(s/t). \tag{4.5}$$

Here

$$\overline{A}(x) = \begin{cases} (\sigma_b^2 + \frac{\overline{K}_1}{n!}(\delta^{<}(t))^{n-1})x, & 0 \leq x \leq \bar{b}, \\ 1 + (\sigma_e^2 - \frac{\overline{K}_2}{2}\delta^{>}(t))(x-1), & \bar{b} < x \leq 1, \end{cases} \tag{4.6}$$

with

$$\bar{b} = \frac{1 - \sigma_e^2 + \frac{\overline{K}_2}{2}\delta^{>}(t)}{\sigma_b^2 + \frac{\overline{K}_1}{n!}(\delta^{<}(t))^{n-1} - \sigma_e^2 + \frac{\overline{K}_2}{2}\delta^{>}(t)}. \tag{4.7}$$

If $\sigma_e^2 < \infty$,

$$\underline{A}(x) = \begin{cases} \left\{(\sigma_b^2 - \frac{\underline{K}_1}{n!}(\delta^{<}(t))^{n-1})x\right\} \vee 0, & 0 \leq x \leq \underline{b}, \\ 1 + (\sigma_e^2 + \frac{\underline{K}_2}{2}\delta^{>}(t))(x-1), & \underline{b} < x \leq 1, \end{cases} \tag{4.8}$$

with

$$\underline{b} = \frac{1 - \sigma_e^2 - \frac{K_2}{2}\delta^>(t)}{\sigma_b^2 - \frac{K_1}{n!}(\delta^<(t))^{n-1} - \sigma_e^2 - \frac{K_2}{2}\delta^>(t)}. \tag{4.9}$$

Remark 4.3. If $\sigma_b^2 = 0$, $\underline{A}(x) = 0$ for $0 \leq x \leq \underline{b}$. If $\lim_{t\uparrow\infty} \delta^<(t) = \delta^< \neq 0$ (which implies that all derivatives in zero are 0), we take

$$\overline{A}(x) = \begin{cases} 0, & 0 \leq x \leq \bar{b}, \\ 1 + (\sigma_e^2 - \frac{\overline{K}_2}{2}\delta^>(t))(x-1), & \bar{b} < x \leq 1, \end{cases} \tag{4.10}$$

and

$$\bar{b} = \frac{1 - \sigma_e^2 + \frac{\overline{K}_2}{2}\delta^>(t)}{-\sigma_e^2 + \frac{\overline{K}_2}{2}\delta^>(t)}. \tag{4.11}$$

If $A'(1) = \sigma_e^2 = +\infty$, then $\underline{b} = 1$. And $\overline{A} \equiv \overline{A}_\rho$ is defined by

$$\overline{A}(x) = \begin{cases} (\sigma_b^2 + \frac{\overline{K}_1}{n!}(\delta^<(t))^{n-1})x, & 0 \leq x \leq \bar{b}, \\ 1 + \rho(x-1), & \bar{b} < x \leq 1, \end{cases} \tag{4.12}$$

and $\bar{b} \equiv \bar{b}_\rho = \frac{1-\rho}{\sigma_b^2 + \frac{\overline{K}_1}{n!}(\delta^<(t))^{n-1} - \rho}$.

The choice of $\overline{\Sigma}^2$ and $\underline{\Sigma}^2$ is motivated by the following properties.

Lemma 4.4. *$\overline{A}$ and $\underline{A}$ are piecewise linear, continuous functions with $\overline{A}(0) = \underline{A}(0) = 0$ and $\overline{A}(1) = \underline{A}(1) = 1$. Moreover,*

(i) *If $\lim_{t\uparrow\infty} \delta^<(t) = 0$, then, for all s with $\Sigma^2(s) \in [0, t^{1/3}]$ and $\Sigma^2(s) \in [t - t^{1/3}, t]$,*

$$\overline{\Sigma}^2(s) \geq \Sigma^2(s) \geq \underline{\Sigma}^2(s). \tag{4.13}$$

(ii) *If $\lim_{t\uparrow\infty} \delta^<(t) = \delta^< > 0$, then (4.13) only holds for all s with $\Sigma^2(s) \in [t - t^{1/3}, t]$ while, for $s \in [0, (\delta \wedge \bar{b})t)$,*

$$\overline{\Sigma}^2(s) = \Sigma^2(s) = \underline{\Sigma}^2(s) = 0. \tag{4.14}$$

Proof: $\overline{A}$ and $\underline{A}$ are obviously piecewise linear. The fact that they are continuous is easily verified. By definition, $A'(0) = \sigma_b^2$ and $A'(1) = \sigma_e^2$. For all s such that $\Sigma^2(s) \in [0, t^{1/3}]$, a nth-order Taylor expansion of $\overline{B}$ at 0 together with the fact that by assumption, for $k < n$, $\overline{B}(0) = \overline{B}^k(0) = 0$ shows that

$$\Sigma^2(s) \leq \overline{B}(s) = t\left[\overline{B}'(0)\frac{s}{t} + \frac{\overline{B}^{(n)}(\xi)}{n!}\left(\frac{s}{t}\right)^n\right], \quad \text{for some } \xi \in (0, s). \tag{4.15}$$

The reverse inequality holds when $\overline{B}$ is replaced by $\underline{B}$. Eq. (4.13) follows then from Assumption (A2). Using a second order Taylor expansion of $\overline{C}$ and $\underline{C}$ at 1, we obtain Eq. (4.13) for $\Sigma^2(s) \in [t - t^{1/3}, t]$. Eq. (4.14) holds trivially in the specified interval. This concludes the proof of the lemma. □

Let $\{\bar{y}_i, i \leq n(t)\}$ be the particles of a BBM with speed function $\overline{\Sigma}^2$ and let $\{\underline{y}_i, i \leq n(t)\}$ be particles of a BBM with speed function $\underline{\Sigma}^2$. We want to show that the limiting extremal processes of these processes coincide. Set

$$\overline{\mathcal{N}}_u(t) \equiv \sum_{i=1}^{n(t)} \mathbb{1}_{\overline{y}_i(t)-\tilde{m}(t)>u}, \tag{4.16}$$

$$\underline{\mathcal{N}}_u(t) \equiv \sum_{i=1}^{n(t)} \mathbb{1}_{\underline{y}_i(t)-\tilde{m}(t)>u}. \tag{4.17}$$

Lemma 4.5. *For all $u_1,\ldots,u_k$ and all $c_1,\ldots,c_k \in \mathbb{R}_+$, the limits*

$$\lim_{t\uparrow\infty} \mathbb{E}\left(\exp\left(-\sum_{l=1}^{k} c_k \overline{\mathcal{N}}_{u_l}(t)\right)\right) \tag{4.18}$$

and

$$\lim_{t\uparrow\infty} \mathbb{E}\left(\exp\left(-\sum_{l=1}^{k} c_k \underline{\mathcal{N}}_{u_l}(t)\right)\right) \tag{4.19}$$

exist. If $A'(1) < \infty$, then two limits coincide with $\mathcal{L}_{u_1,\ldots,u_k}(c)$.

If $A'(1) = \sigma_e^2 = \infty$, then the two limits in (4.18) and (4.19) converges to the same limit, as $\rho \uparrow \infty$.

Proof: We first consider the case when $A'(1) < \infty$. To prove Lemma 4.5, we show that the extremal processes

$$\overline{\mathcal{E}_t} = \sum_{i=1}^{n(t)} \delta_{\overline{y}_i - \tilde{m}(t)} \quad \text{and} \quad \underline{\mathcal{E}_t} = \sum_{i=1}^{n(t)} \delta_{\underline{y}_i - \tilde{m}(t)} \tag{4.20}$$

both converge to $\mathcal{E}_{\sigma_b,\sigma_e}$, that was defined in (1.13). Note that this implies first convergence of Laplace functionals with functions ϕ with compact support, while the $\mathcal{N}_u(t)$ have support that is unbounded from above. Convergence for these, however, carries over due to the tightness established in Proposition 3.1.

To do so, observe that the slopes at 0 of $\overline{\Sigma}^2$ and $\underline{\Sigma}^2$ are equal to σ_b^2 up to an error of order $\delta^<(t)$. Moreover, the slope at t is in both cases, up to an error of order $\delta^>(t)$, equal to σ_e^2. The time of speed change $\overline{b}(t)$, respectively $\underline{b}(t)$, is equal to $\frac{1-\sigma_e^2}{\sigma_b^2-\sigma_b^2}$ up to an error of order $\delta^>(t) \vee \delta^<(t)$. For the two-speed BBM with speed

$$\sigma^2(s) = \begin{cases} \sigma_b^2, & \text{for } 0 < s \le \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2}, \\ \sigma_e^2, & \text{for } \frac{1-\sigma_e^2}{\sigma_b^2-\sigma_e^2} < s < t, \end{cases} \tag{4.21}$$

it was shown in Bovier and Hartung (2014) that the maximal displacement is equal to $\tilde{m}(t)$ and that the extremal process converges to $\mathcal{E}_{\sigma_b,\sigma_e}$ as $t \uparrow \infty$. The method used to show this is to show the convergence of the Laplace functionals, $\mathbb{E}(\exp(-\int \phi(x)\mathcal{E}_t(dx)))$, $\phi \in C_c(\mathbb{R},\mathbb{R}_+)$. The other difference is that the function A we have to consider now depend (weakly) on t. We need to show that this has no effect.

Inspecting the proof of the convergence of the Laplace functional, respectively convergence of the maximum in Bovier and Hartung (2014), one sees that nothing changes (since we keep t fixed) until Eq. (5.28) in Bovier and Hartung (2014).

There, we then have to show that, for each $y \in \mathbb{R}$, (in the case of $\overline{\Sigma}^2$)

$$\mathbb{E}\left(\exp\left(-C(a)\left(\frac{\sigma_e^2-\frac{\overline{K}_2}{2}\delta^>(t)}{1-[\sigma_b^2+\frac{\overline{K}_1}{2}\delta^<(t)]\bar{b}/\sqrt{t}}\right)^{1/2} \mathrm{e}^{-\sqrt{2}y}\overline{Y}^B_{\sigma_b,\bar{b}\sqrt{t},\gamma}\right)(1+o(1))\right), \tag{4.22}$$

converges, as first $t \uparrow \infty$ and then $B \uparrow \infty$, to

$$\mathbb{E}\left(\exp\left(-\sigma_e C(a) Y_{\sigma_b}\mathrm{e}^{-\sqrt{2}y}\right)\right), \tag{4.23}$$

where $C(a) > 0$ is a constant depending on $a = \sqrt{2}(\sigma_e - 1)$ (see (1.15)), and

$$\begin{aligned}&\overline{Y}^B_{\sigma_b,\bar{b}\sqrt{t},\gamma} \\ &= \sum_{i=1}^{n(\bar{b}\sqrt{t})} \mathrm{e}^{-(1+\sigma_b^2+\frac{K}{2}\delta^<(t))\bar{b}\sqrt{t}+\sqrt{2}\overline{y}_i(\bar{b}\sqrt{t})} \mathbb{1}_{\overline{y}_i(\bar{b}\sqrt{t})-\sqrt{2}(\sigma_b^2+\frac{K}{2}\delta^<(t))\bar{b}\sqrt{t}\in[-Bt^{\gamma/2},Bt^{\gamma/2}]}.\end{aligned} \tag{4.24}$$

The main task is to ensure the convergence of $\overline{Y}^B_{\sigma_b,\bar{b}\sqrt{t},\gamma}$ to the limit of the corresponding McKean martingale, Y_{σ_b}. In the case where $\lim_{t\uparrow\infty}\delta^<(t) > 0$, this takes the simple form

$$\overline{Y}^B_{0,\bar{b}\sqrt{t},\gamma} = \sum_{i=1}^{n(\bar{b}\sqrt{t})} \mathrm{e}^{-\bar{b}\sqrt{t}}, \tag{4.25}$$

which converges to an exponential random variable of mean one, as desired.

In the case when $\lim_{t\uparrow\infty}\delta^<(t) = 0$, a further slight modification is necessary. Observe that in the proof of Theorem 5.1 in Bovier and Hartung (2014), $\bar{b}\sqrt{t}$ can be replaced by any sequence $\Delta(t) \uparrow \infty$ such that $\lim_{t\uparrow\infty}\Delta(t)/t = 0$. Here we adapt $\Delta(t)$ to the function Σ^2 and choose

$$\Delta(t) = \left(\delta^<(t)\right)^{-1/2}. \tag{4.26}$$

Doing so, we have to show that, analogously to (4.22), the object

$$\mathbb{E}\left(\exp\left(-C(a)\left(\frac{\sigma_e^2-\frac{\overline{K}_2}{2}\delta^>(t)}{1-[\sigma_b^2+\frac{\overline{K}_1}{2}\delta^<(t)]\Delta(t)/t}\right)^{1/2} \mathrm{e}^{-\sqrt{2}y}\overline{Y}^B_{\sigma_b,\Delta(t),\gamma}\right)(1+o(1))\right) \tag{4.27}$$

converges to (4.23). By our choice of $\Delta(t)$,

$$\left|\mathrm{e}^{-\frac{\overline{K}_1}{2}\delta^<(t)\Delta(t)}\mathrm{e}^{\sqrt{2}\frac{\overline{K}_1}{2}\delta^<(t)(\Delta(t)+B\Delta(t)^\gamma)}-1\right| \leq const.\sqrt{\delta^<(t)}, \tag{4.28}$$

which tends to zero, as $t \uparrow \infty$. Thus

$$\overline{Y}^B_{\sigma_b,\Delta(t),\gamma} = \widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma}(1+o(1)), \tag{4.29}$$

where

$$\widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma} \equiv \sum_{i=1}^{n(\Delta(t))} \mathrm{e}^{-(1+\sigma_b^2)\Delta(t)+\sqrt{2}\sigma_b\overline{x}_i(\Delta(t))} \mathbb{1}_{\sigma_b\overline{x}_i(\Delta(t))-\sqrt{2}\sigma_b^2\Delta(t)\in[-B\Delta(t)^\gamma,B\Delta(t)^\gamma]}. \tag{4.30}$$

By Lemma 4.3 in Bovier and Hartung (2014), it follows that $\widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma}$ converges in probability and in L^1 to the random variable Y_{σ_b}. Since $\widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma} \geq 0$ and $C(a) > 0$, and since

$$\lim_{t\uparrow\infty}\left(\frac{\sigma_e^2-\frac{\overline{K}_2}{2}\delta^>(t)}{1-[\sigma_b^2+\frac{\overline{K}_1}{2}\delta^<(t)]\Delta(t)/t}\right)^{1/2} = \sigma_e, \tag{4.31}$$

it follows that

$$\begin{aligned}
&\lim_{B\uparrow\infty}\liminf_{t\uparrow\infty}\mathbb{E}\left(\exp\left(-C(a)\sigma_e \mathrm{e}^{-\sqrt{2}y}\widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma}\right)(1+o(1))\right)\\
&=\lim_{B\uparrow\infty}\limsup_{t\uparrow\infty}\mathbb{E}\left(\exp\left(-C(a)\sigma_e \mathrm{e}^{-\sqrt{2}y}\widetilde{Y}^B_{\sigma_b,\Delta(t),\gamma}\right)(1+o(1))\right)\\
&=\mathbb{E}\left(\exp\left(-\widetilde{C}(\sigma_e)Y_{\sigma_b}\mathrm{e}^{-\sqrt{2}y}\right)\right), \qquad (4.32)
\end{aligned}$$

where $\widetilde{C}(\sigma_e)=\sigma_e C(a)$. The same arguments work when $\overline{\Sigma}^2$ is replaced by $\underline{\Sigma}^2$. The limit in (4.32) coincides with the one obtained in Bovier and Hartung (2014) for the two-speed BBM with speed given in (4.21). The assertion in the case when $\sigma_e=\infty$ follows directly from Lemma 3.3. □

4.2. *Gaussian comparison.* We distinguish from now on the expectation with respect to the underlying tree structure and the one with respect to the Brownian movement of the particles.

- $\mathbb{E}_n$: expectation w.r.t. Galton-Watson process.
- $\mathbb{E}_B$: expectation w.r.t. the Gaussian process conditioned on the σ-algebra $\mathcal{F}_t^{\mathrm{tree}}$ generated by the Galton-Watson process.

For a given realisation of the Galton-Watson process, we now let x, $\bar{y}$, and $\underline{y}$ be three independent Gaussian processes whose covariances are given as in (1.2) with A replaced by $\overline{A}$ in the case of $\bar{y}$ and $\underline{A}$ in the case of $\underline{y}$.

The proof of Proposition 4.1 is based on the following Lemma that compares the Laplace transform $\mathcal{L}_{u_1,\dots,u_k}(t,c)$ with the corresponding Laplace transform for the comparison processes.

Lemma 4.6. *For any $k\in\mathbb{N}$, $u_1,\dots,u_k\in\mathbb{R}$ and $c_1,\dots,c_k\in\mathbb{R}_+$ we have*

$$\mathcal{L}_{u_1,\dots,u_k}(t,c) \;\leq\; \mathbb{E}\left(\exp\left(-\sum_{l=1}^k c_l\overline{\mathcal{N}}_{u_l}(t)\right)\right)+o(1) \qquad (4.33)$$

$$\mathcal{L}_{u_1,\dots,u_k}(t,c) \;\geq\; \mathbb{E}\left(\exp\left(-\sum_{l=1}^k c_l\underline{\mathcal{N}}_{u_l}(t)\right)\right)+o(1) \qquad (4.34)$$

Proof: The proofs of (4.33) and (4.34) are very similar. Hence we focus on proving (4.33). We will, however, indicate what has to be changed when proving the lower bound as we go along. For simplicity all overlined names depend on $\overline{\Sigma}^2$. Corresponding quantities where $\overline{\Sigma}^2$ is replaced by $\underline{\Sigma}^2$ are underlined.

To use Gaussian comparison methods, we first replace the functions $\mathcal{N}_u(t),\overline{\mathcal{N}}_u(t)$ by smooth approximants:

$$\chi^\kappa(x)\equiv\frac{1}{\sqrt{2\pi\kappa^2}}\int_{-\infty}^x \mathrm{e}^{-z^2/2\kappa^2}dz, \qquad (4.35)$$

$$\mathcal{N}_u^\kappa(t)\equiv\sum_{i=1}^{n(t)}\chi^\kappa(x_i(t)-\tilde{m}(t)-u), \qquad (4.36)$$

and

$$\overline{\mathcal{N}}_u^\kappa(t)\equiv\sum_{i=1}^{n(t)}\chi^\kappa(\bar{y}_i(t)-\tilde{m}(t)-u). \qquad (4.37)$$

Note that, as $\kappa \downarrow 0$,

$$\chi^\kappa(x) \to \mathbb{1}_{x>0}, \quad \mathcal{N}_u^\kappa(t) \to \mathcal{N}_u(t), \quad \overline{\mathcal{N}}_u^\kappa(t) \to \overline{\mathcal{N}}_u(t). \tag{4.38}$$

In order to prove (4.33), we show that for all $\kappa > 0$,

$$\mathbb{E}_B\left(\exp\left(-\sum_{l=1}^k c_l \mathcal{N}_{u_l}^\kappa(t)\right)\right) \le \mathbb{E}_B\left(\exp\left(-\sum_{l=1}^k c_l \overline{\mathcal{N}}_{u_l}^\kappa(t)\right)\right) + R(t), \tag{4.39}$$

where $R(t)$ is independent of κ and $\lim_{t\uparrow\infty} \mathbb{E}R(t) = 0$.

From now on we work conditional on the σ-algebra generated by the Galton-Watson tree. We introduce the family of functions $f_{t,\kappa} : \mathbb{R}^{n(t)} \to \mathbb{R}$ by

$$f_{t,\kappa}(x) \equiv f_{t,\kappa}(x_1, \dots, x_{n(t)}) \equiv \exp\left(-\sum_{i=1}^{n(t)} \sum_{l=1}^k c_l \chi^\kappa(x_i - \tilde{m}(t) - u_l)\right).$$

We want to control

$$\mathbb{E}_B\left(\exp\left(-\sum_{l=1}^k c_l \mathcal{N}_{u_l}^\kappa(t)\right)\right) - \mathbb{E}_B\left(\exp\left(-\sum_{l=1}^k c_l \overline{\mathcal{N}}_{u_l}^\kappa(t)\right)\right)$$
$$= \mathbb{E}_B\left(f_{t,\kappa}(x_1(t), \dots, x_{n(t)}(t))\right) - \mathbb{E}_B\left(f_{t,\kappa}(\overline{y}_1(t), \dots, \overline{y}_{n(t)}(t))\right) \tag{4.40}$$

Define for $h \in [0,1]$ the interpolating process

$$x_i^h = \sqrt{h} x_i + \sqrt{1-h}\,\overline{y}_i, \quad h \in [0,1]. \tag{4.41}$$

The interpolating process $\{x_i^h, i \le n(t)\}$ is a Gaussian process with the same underlying branching structure and speed function

$$\Sigma_h^2(s) = h\Sigma^2(s) + (1-h)\overline{\Sigma}^2(s). \tag{4.42}$$

Then, (4.40) is equal to

$$\mathbb{E}_B\left(\int_0^1 \frac{d}{dh} f_{t,\kappa}(x^h(t)) dh\right), \tag{4.43}$$

where

$$\frac{d}{dh} f_{t,\kappa}(x^h(t)) = \frac{1}{2} \sum_{i=1}^{n(t)} \frac{\partial}{\partial x_i} f_{t,\kappa}(x_1^h(t), \dots, x_{n(t)}^h(t)) \left[\frac{1}{\sqrt{h}} x_i(t) - \frac{1}{\sqrt{1-h}} \overline{y}_i(t)\right], \tag{4.44}$$

and

$$\frac{\partial}{\partial x_i} f_{t,\kappa}(x_1^h(t), \dots, x_{n(t)}^h(t)) \tag{4.45}$$
$$= -\left(\sum_{l=1}^k \frac{c_l}{\sqrt{2\pi\kappa^2}} \mathrm{e}^{-\frac{(x_i^h(t) - \tilde{m}(t) - u_l)^2}{2\kappa^2}}\right) f_{t,\kappa}(x_1^h(t), \dots, x_{n(t)}^h(t)).$$

The key idea is to introduce a localisation condition on the path of x_i^h into (4.44) at this stage. Note that it is not surprising at this point, since localising paths has been a crucial tool in almost all computations involving BBM, see already Bramson's paper Bramson (1978a). To do so, we insert into the right-hand side of (4.44) a one in the form

$$1 = \mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} + \mathbb{1}_{x_i^h \notin \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}}, \tag{4.46}$$

with

$$\bar{I} \equiv \left[t(\delta_0^<(t) \wedge \delta_1^<(t)), t(1-\delta_1^>(t))\right], \tag{4.47}$$

and $\mathcal{T}^\gamma_{t,I,\Sigma_h^2}$ defined in (2.1). Here $\delta_1^{<,>}(t) \equiv \delta^{<,>}(t)$, while $\delta_0^{<,>}$ is defined in the same way, but with respect to the speed function $\overline{\Sigma}^2$ instead of Σ^2. We call the two resulting summands $\overline{S}^h_<$ and $\overline{S}^h_>$, respectively.

Note that, when proving the lower bound, we choose instead of $\bar{I}$, the interval

$$\underline{I} \equiv \left[t(\delta_0^<(t) \wedge \delta_1^<(t)), t(1-\delta_0^>(t))\right]. \tag{4.48}$$

The next lemma shows that $\overline{S}^h_>$ does not contribute to the expectation in (4.44), as $t \to \infty$.

Lemma 4.7. *With the notation above, we have*

$$\lim_{t\to\infty} \mathbb{E}_n \left(\int_0^1 \mathbb{E}_B(|\overline{S}^h_>|) dh \right) = 0. \tag{4.49}$$

The proof of this lemma will be postponed.

We continue with the proof of Lemma 4.6. We are left with controlling, for fixed $h \in (0,1)$,

$$\mathbb{E}_B(\overline{S}^h_<) = \mathbb{E}_B \left(\frac{1}{2} \sum_{i=1}^{n(t)} \frac{\partial}{\partial x_i} f_{t,\kappa}(x^h(t)) \mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \left[\frac{x_i(t)}{\sqrt{h}} - \frac{\overline{y}_i(t)}{\sqrt{1-h}} \right] \right). \tag{4.50}$$

By the definition of $\mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}$,

$$\mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} = \mathbb{1}_{\forall s \in \bar{I}: \left|\xi_i^h(s)\right| \leq (\Sigma_h^2(s) \wedge (t - \Sigma_h^2(s)))^\gamma}, \tag{4.51}$$

where $\xi_i^h(s) \equiv x_i^h(s) - \frac{\Sigma_h^2(s)}{t} x_i^h(t)$ is a time changed Brownian bridge from 0 to 0 in time t, which, as we recall, is independent of the endpoint $x_i^h(t)$. We want to apply a Gaussian integration by parts formula to (4.50). However, we need to take care of the fact that each summand in (4.50) depends on the whole path of ξ_i through the term in (4.51). Therefore, we first approximate that indicator function in (4.51) by a discretised version. Let, for $N \in \mathbb{N}$, $t_1, \ldots, t_{2^N}$ be a sequence of equidistant points in $[0,t]$. Define the following sequence of approximations, $G_{h,N} : C(\mathbb{R}_+) \to \mathbb{R}$, to the indicator function in (4.51),

$$G_{h,N}(x) \equiv g_h(x(t_1), \ldots, x(t_{2^N})), \tag{4.52}$$

where

$$\begin{aligned} g_h(z_1, \ldots, z_{2^N}) &= \prod_{\ell=1}^{2^N} \Big[\mathbb{1}_{t_\ell \in \bar{I}} \chi^{2^{-N}} \left((\Sigma_h^2(t_\ell) \wedge (t - \Sigma_h^2(t_\ell)))^\gamma - z_\ell \right) \\ &\times \chi^{2^{-N}} \left((\Sigma_h^2(t_\ell) \wedge (t - \Sigma_h^2(t_\ell)))^\gamma + z_\ell \right) + \mathbb{1}_{t_\ell \notin \bar{I}} \Big]. \end{aligned} \tag{4.53}$$

Clearly $G_{h,N} \to \mathbb{1}_{x \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}}$, pointwise, while the derivatives $\frac{\partial}{\partial z_\ell} g_h(z_1, \ldots, z_{2^N})$ are bounded. By the Gaussian integration by parts formula (see, e.g., Talagrand (2011a,

Appendix A.5)), we have, for any given $N \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}_B\left(x_i(t)\frac{\partial}{\partial x_i} f_{t,\kappa}(x^h(t)) G_{h,N}(\xi^h)\right)\\
&= \sum_{\ell=1}^{2^N} \mathbb{E}_B\left((x_i(t)\xi_i^h(t_\ell)\right) \mathbb{E}_B\left(f_{t,\kappa}(x^h(t))\frac{\partial}{\partial z_\ell} g_h(\xi_i^h(t_1),\dots,\xi_i^h(t_{2^N}))\right)\\
&\quad + \sum_{j=1}^{n(t)} \mathbb{E}_B(x_i(t)x_j^h(t))\mathbb{E}_B\left(G_{h,N}(\xi^h)\frac{\partial^2}{\partial x_j \partial x_i} f_{t,\kappa}(x^h(t))\right). \qquad (4.54)
\end{aligned}
$$

But for all $\ell \in \{1,\dots,2^N\}$,

$$
\begin{aligned}
\mathbb{E}_B\left(x_i(t)\xi_i^h(t_\ell)\right) &= \sqrt{h}\mathbb{E}_B\left(x_i(t)x_i(t_\ell) - x_i(t)\frac{\Sigma^2(t_\ell)}{t}x_i(t)\right)\\
&= \sqrt{h}\left(\Sigma^2(t_\ell) - \Sigma^2(t_\ell)\right) = 0, \qquad (4.55)
\end{aligned}
$$

and hence the second line in (4.54) is equal to zero. In exactly the same way we get

$$
\begin{aligned}
&\mathbb{E}_B\left(\bar{y}_i \frac{\partial}{\partial x_i} f_{t,\kappa}(x_1^h(t),\dots,x_{n(t)}^h(t))\right)\\
&= \sum_{j=1}^{n(t)} \mathbb{E}_B(\bar{y}_i(t)x_j^h(t))\mathbb{E}_B\left(G_{h,N}(\xi^h)\frac{\partial^2}{\partial x_j\partial x_i} f_{t,\kappa}(x^h(t))\right). \qquad (4.56)
\end{aligned}
$$

Computing the covariances, $\mathbb{E}_B\left(x_i(t)x_j^h(t)\right) = \sqrt{h}\mathbb{E}\left(x_i(t)x_j(t)\right)$ and

$$
\mathbb{E}_B\left(\bar{y}_i(t)x_j^h(t)\right) = \sqrt{1-h}\mathbb{E}\left(\bar{y}_i(t)\bar{y}_j(t)\right),
$$

we obtain that

$$
\begin{aligned}
&\mathbb{E}_B\left(\frac{1}{2}\sum_{i=1}^{n(t)} \frac{\partial}{\partial x_i} f_{t,\kappa}(x^h(t)) G_{h,N}(\xi^h)\left[\frac{x_i(t)}{\sqrt{h}} - \frac{\overline{y}_i(t)}{\sqrt{1-h}}\right]\right) \qquad (4.57)\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left[\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t))\right] \mathbb{E}_B\left(G_{h,N}(\xi^h)\frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right),
\end{aligned}
$$

where crucially the terms with $i = j$ have cancelled. This equation holds for any N, and since $0 \le G_{h,N}(x) \le 1$, and the integral $\mathbb{E}_B\left(\frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right)$ is finite (trivially, since the mixed second derivatives of f are bounded), by Lebesgue's dominated convergence theorem, the right hand side converges to the expression where $G_{h,N}$ is replaced by its limit. Similarly, in the left hand side we can apply Lebesgue's theorem, majorising the integrands by $C|x_i(t)|$, etc, which are all integrable. Thus we obtain that

$$
\begin{aligned}
&\mathbb{E}_B\left(\frac{1}{2}\sum_{i=1}^{n(t)} \frac{\partial}{\partial x_i} f_{t,\kappa}(x^h(t)) \mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \left[\frac{x_i(t)}{\sqrt{h}} - \frac{\overline{y}_i(t)}{\sqrt{1-h}}\right]\right) \qquad (4.58)\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left[\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t))\right] \mathbb{E}_B\left(\mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right),
\end{aligned}
$$

Introducing

$$1 = \mathbb{1}_{d(x_i^h(t),x_j^h(t))\in \bar{I}} + \mathbb{1}_{d(x_i^h(t),x_j^h(t))\notin \bar{I}}, \tag{4.59}$$

into (4.58), we rewrite the right hand side of (4.58) as $(\overline{T1}) + (\overline{T2})$, where

$$(\overline{T1}) \tag{4.60}$$
$$= \sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \mathbb{E}_B\left[x_i(t)x_j(t) - \overline{y}_i(t)\overline{y}_j(t)\right] \mathbb{E}_B\left(\mathbb{1}_{d(x_i^h(t),x_j^h(t))\in \bar{I}} \mathbb{1}_{x_i^h\in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right),$$

$$(\overline{T2}) \tag{4.61}$$
$$= \sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \mathbb{E}_B\left[x_i(t)x_j(t) - \overline{y}_i(t)\overline{y}_j(t)\right] \mathbb{E}_B\left(\mathbb{1}_{d(x_i^h(t),x_j^h(t))\notin \bar{I}} \mathbb{1}_{x_i^h\in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right).$$

The term $(\overline{T1})$ is controlled by the following Lemma.

Lemma 4.8. *With the notation above, there exists a constant $\widetilde{C} < \infty$, independent of t and κ^2, such that for all t large and κ^2 small enough,*

$$\left|\mathbb{E}_n\left(\int_0^1 (\overline{T1})dh\right)\right| \leq \widetilde{C}\int_{\bar{I}} \left|\mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)}\right| ds. \tag{4.62}$$

Moreover, we have:

Lemma 4.9. *If Σ^2 satisfies (A1)-(A3), and $\overline{\Sigma}^2$ is as defined in (4.4), then*

$$\lim_{t\to\infty}\int_{\bar{I}} \left|\mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)}\right| ds = 0. \tag{4.63}$$

We postpone the proofs of these lemmata to Section 5.

Up to this point the proof of (4.34) works exactly as the proof of (4.33) when $\overline{\Sigma}^2$ is replaced by $\underline{\Sigma}^2$. For $(\overline{T2})$ and $(\underline{T2})$ we have:

Lemma 4.10. *For almost all realisations of the Galton-Watson process, the following statements hold:*

(i) *If $\lim_{t\uparrow\infty}\delta^<(t) = 0$, then*

$$(\overline{T2}) \leq 0, \tag{4.64}$$

and

$$(\underline{T2}) \geq 0. \tag{4.65}$$

(ii) *If $\lim_{t\uparrow\infty}\delta^<(t) = \delta^< > 0$, then*

$$\lim_{t\uparrow\infty}(\overline{T2}) \leq 0, \tag{4.66}$$

and

$$\lim_{t\uparrow\infty}(\underline{T2}) \geq 0. \tag{4.67}$$

The proof of this lemma is again postponed to Section 5.

From Lemma 4.8, Lemma 4.9, and Lemma 4.10 together with (4.50), the bound (4.39) follows. Since the left and right hand sides involve expectations over bounded functions, we may pass to the limit $\kappa^2 \downarrow 0$. This implies (4.33). As pointed out, using Lemma 4.10, the bound (4.34) also follows. Thus, Lemma 4.6 is proved, once we provide the postponed proofs of the various lemmata above. □

We conclude the proof of Proposition 4.1.

Proof of Proposition 4.1: Taking the limit as $t \uparrow \infty$ in (4.33) and (4.34) and using Lemma 4.5 gives, in the case $A'(1) < \infty$,

$$\limsup_{t\uparrow\infty} \mathcal{L}_{u_1,\dots,u_k}(t,c) \quad \le \quad \mathcal{L}_{u_1,\dots,u_k}(c), \tag{4.68}$$

$$\liminf_{t\uparrow\infty} \mathcal{L}_{u_1,\dots,u_k}(t,c) \quad \ge \quad \mathcal{L}_{u_1,\dots,u_k}(c). \tag{4.69}$$

Hence $\lim_{t\uparrow\infty} \mathcal{L}_{u_1,\dots,u_k}(t,c)$ exists and is equal to $\mathcal{L}_{u_1,\dots,u_k}(c)$. In the case $A'(1) = \infty$, the same result follows if in addition we take $\rho \uparrow \infty$ after taking $t \uparrow \infty$. This concludes the proof of Proposition 4.1. $\square$

5. Proofs of the auxiliary lemmata

We now provide the proofs of the lemmata from the last section whose proofs we had postponed.

Proof of Lemma 4.7: We have

$$\mathbb{E}_B(|\overline{S}^h_>|) \le \frac{1}{2}\sum_{i=1}^{n(t)}\sum_{l=1}^{k} c_l \mathbb{E}_B\left(\frac{\mathrm{e}^{-\frac{(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}}}{\sqrt{2\pi\kappa^2}} \mathbb{1}_{x_i^h \not\in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}}\left[\frac{|x_i(t)|}{\sqrt{h}} + \frac{|\overline{y}_i(t)|}{\sqrt{1-h}}\right]\right). \tag{5.1}$$

We use the fact that the condition in the indicator function involves only the time changed Brownian bridge, $\xi_i^h(s) = x_i^h(s) - \frac{\Sigma_h^2(s)}{t} x_i^h(t)$, which is independent of the endpoint $x_i^h(t)$, and of course also of $x_i(t)$. This implies that

$$\mathbb{E}_B\left(\frac{\mathrm{e}^{-\frac{(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}}}{\sqrt{2\pi\kappa^2}} \mathbb{1}_{x_i^h \not\in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \frac{|x_i(t)|}{\sqrt{h}}\right)$$

$$= \mathbb{E}_B\left(\frac{\mathrm{e}^{-\frac{(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}}}{\sqrt{2\pi\kappa^2}} \frac{|x_i(t)|}{\sqrt{h}}\right)\mathbb{P}_B\left(x_i^h \not\in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}\right), \tag{5.2}$$

and similarly for the terms involving $\bar{y}$. The computation of the first expectation is a straightforward Gaussian integration involving two independent Gaussians. In fact we can write

$$\mathbb{E}_B\left(\frac{\mathrm{e}^{-\frac{(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}}}{\sqrt{2\pi\kappa^2}}|x_i(t)|\right) = \int \frac{dz_1 dz_2}{(2\pi)^{3/2} t\kappa} \mathrm{e}^{-\frac{1}{2}(\underline{z}, M\underline{z}) + (\underline{v},\underline{z}) - (\tilde{m}(t)+u_l)^2/2\kappa^2}|z_1|, \tag{5.3}$$

where

$$M \equiv \begin{pmatrix} \frac{\kappa^2+th}{t\kappa^2} & \sqrt{h(1-h)}/\kappa^2 \\ \sqrt{h(1-h)}/\kappa^2 & \frac{\kappa^2+t(1-h)}{t\kappa^2}\end{pmatrix}, \qquad \underline{v} \equiv \frac{\tilde{m}(t)+u_l}{\kappa^2}\begin{pmatrix}\sqrt{h} \\ \sqrt{1-h}\end{pmatrix}. \tag{5.4}$$

Note that $\det M = t^{-2} + t^{-1}\kappa^{-2}$, and its eigenvalues are given by

$$\lambda_\pm = t^{-1} + \kappa^{-2} \pm \sqrt{\kappa^{-4} + t^{-1}\kappa^{-2}}. \tag{5.5}$$

Importantly, the smaller eigenvalue behaves, when κ^2/t tends to zero, as

$$\lambda_- = 1/(2t)\left(1 + O(\kappa^2/t)\right). \tag{5.6}$$

The remaining calculations amount to completing the square. With

$$\underline{a} \equiv \frac{\tilde{m}(t) + u_l}{\kappa^2 t^{-1} + 1} \begin{pmatrix} \sqrt{h} \\ \sqrt{1-h} \end{pmatrix}, \tag{5.7}$$

we can rewrite the right hand side of (5.3) as

$$\frac{\mathrm{e}^{-\frac{1}{2}\frac{(\tilde{m}(t)+u_l)^2}{t+\kappa^2}}}{(2\pi)^{1/2}\sqrt{\kappa^2+t}} \int \frac{dz_1 dz_2}{2\pi\sqrt{\det(M^{-1})}} \mathrm{e}^{-\frac{1}{2}(\underline{z}-\underline{a}, M(\underline{z}-\underline{a}))^2} |z_1|. \tag{5.8}$$

Now it is plain that the last expectation is bounded by

$$|\underline{a}_1| + const.(\lambda_-)^{-1/2} \leq \sqrt{h}\frac{\tilde{m}(t) + u_l}{\kappa^2/t + 1} + 2t^{1/2}(1 + O(\kappa^2 t^{-2})) \leq const.(\sqrt{h}t + \sqrt{t}), \tag{5.9}$$

with the constant uniform in, say, $\kappa^2 \leq 1, t \geq 100$. This allows us to bound (5.8) by a uniform constant times

$$\left(\sqrt{h}t + 2\right) \mathrm{e}^{-t/(1+\kappa^2/t)+\ln t/(1+\kappa^2/t)} \leq const.\mathrm{e}^{-t}\left(\sqrt{ht^3} + 2t\right). \tag{5.10}$$

Next we bound the probability that the Brownian bridge does not stay in the tube. For this we use Lemma 2.2. Note that by construction, if $s \in \bar{I}$, then for all $h \in [0,1]$, $\Sigma_h^2 \geq Dt^{1/3}$, and $\Sigma_h^2 \leq t - Dt^{1/3}$, for some constant $0 < D < \infty$, depending only on the function A. Thus, by Eq. (2.4)) of Lemma 2.2,

$$\mathbb{P}_B\left(x_i^h \notin \mathcal{T}^{\gamma}_{t,\bar{I},\Sigma_h^2}\right) \leq 8 \sum_{k=Dt^{1/3}}^{\infty} k^{1/2-\gamma}\mathrm{e}^{-k^{2\gamma-1}/2}. \tag{5.11}$$

We are now ready to insert everything back into (5.1). This gives that, uniformly in κ^2 small and t large (as above)

$$\mathbb{E}_B\left(|\overline{S}^h_>|\right) \leq n(t)const.\sum_{l=1}^{k} c_l \mathrm{e}^{-t}\left(2\sqrt{t^3} + 2t/\sqrt{h} + 2t/\sqrt{1-h}\right)\mathrm{e}^{-D^{2\gamma-1}t^{(2\gamma-1)/3}}. \tag{5.12}$$

Integrating over h and taking the expectation with respect to the Galton-Watson process yields

$$\mathbb{E}_n\left(\int_0^1 \mathbb{E}_B\left(|\overline{S}^h_>|\right) dh\right) \leq const.\sum_{l=1}^{k} c_l t^{3/2}\mathrm{e}^{-D^{2\gamma-1}t^{(2\gamma-1)/3}},$$

which tends to zero as $t \uparrow \infty$, uniformly in $\kappa \leq 1$, as claimed, if $\gamma > 1/2$. This proves the assertion of the lemma. □

Proof of Lemma 4.10: We first proof (4.64). Observe that

$$d(x_i(t), x_j(t)) = d(\overline{y}_i(t), \overline{y}_j(t)) = d(x_i^h(t), x_j^h(t)). \tag{5.13}$$

Moreover, for all $1 \leq i, j \leq n(t), i \neq j$,

$$\mathbb{1}_{x_i^h \in \mathcal{T}^{\gamma}_{t,\bar{I},\Sigma_h^2}} \frac{\partial}{\partial x_i \partial x_j} f_{t,\kappa}(x_1^h(t), \ldots, x_{n(t)}^h(t)) \geq 0. \tag{5.14}$$

For $d(x_i(t), x_j(t)) \in [0, t(\delta_1^<(t) \wedge \delta_0^<(t)))$, we distinguish the cases $\lim_{t\to\infty}\delta^<(t) > 0$ and $\lim_{t\to\infty}\delta^<(t) = 0$, respectively.

If $\lim_{t\to\infty}\delta^<(t) = \delta^< > 0$, then $A(x) = \overline{A}(x) = \underline{A}(x) = 0$, for all $x \in [0, t(\delta_1^<(t) \wedge \delta_0^<(t)))$. Thus all the terms in both $(\overline{T2})$ and $(\underline{T2})$ with i, j such that $d(x_i(t), x_j(t)) \in [0, t(\delta_1^<(t) \wedge \delta_0^<(t)))$ vanish.

Next consider the case where $\lim_{t\to\infty}\delta^<(t) = 0$. By Lemma 4.4 we have, for $\overline{y}_i(t), \overline{y}_j(t)$ with $d(\overline{y}_i(t), \overline{y}_j(t)) \in [0, t(\delta_1^<(t) \wedge \delta_0^<(t)))$, that

$$\begin{aligned}
\mathbb{E}_B(\overline{y}_i(t), \overline{y}_j(t)) &= \overline{\Sigma}^2\left(d(\overline{y}_i(t), \overline{y}_j(t))\right) \\
&\geq \Sigma^2(d(\overline{y}_i(t), \overline{y}_j(t))) \\
&= \Sigma^2(d(x_i(t), x_j(t))) = \mathbb{E}_B(x_i(t), x_j(t)). \qquad (5.15)
\end{aligned}$$

For (4.65) we proceed in the same way but instead of (5.15) we have, for $d(\underline{y}_i(t), \underline{y}_j(t)) \in [0, t(\delta_1^<(t) \wedge \delta_0^<(t)))$,

$$\begin{aligned}
\mathbb{E}_B(\underline{y}_i(t), \underline{y}_j(t)) &= \underline{\Sigma}^2\left(d(\underline{y}_i(t), \underline{y}_j(t))\right) \\
&\leq \Sigma^2(d(\underline{y}_i(t), \underline{y}_j(t))) \\
&= \Sigma^2(d(x_i(t), x_j(t))) = \mathbb{E}_B(x_i(t), x_j(t)). \qquad (5.16)
\end{aligned}$$

If $d(\overline{y}_i(t), \overline{y}_j(t)) \in [t(1-\delta_1^>(t)), t]$, resp. $d(\underline{y}_i(t), \underline{y}_j(t)) \in [t(1-\delta_0^>(t)), t]$, we obtain in both cases from Lemma 4.4 that

$$\mathbb{E}_B(\overline{y}_i(t), \overline{y}_j(t)) \geq \mathbb{E}_B(x_i(t), x_j(t)), \qquad (5.17)$$

and

$$\mathbb{E}_B(\underline{y}_i(t), \underline{y}_j(t)) \leq \mathbb{E}_B(x_i(t), x_j(t)), \qquad (5.18)$$

respectively. This concludes the proof of Lemma 4.10. □

Proof of Lemma 4.8 : We have that

$$\begin{aligned}
\left|\mathbb{E}_n\left(\int_0^1 (\overline{T1})dh\right)\right| &\leq \mathbb{E}_n\Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left|\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t))\right| \qquad (5.19) \\
&\times \int_0^1 \mathbb{E}_B\left(\mathbb{1}_{d(x_i^h(t), x_j^h(t))\in \bar{I}}\mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j}\right) dh\Bigg).
\end{aligned}$$

By definition of $f_{t,\kappa}$ we have for $i \neq j$

$$\begin{aligned}
\frac{\partial^2 f_{t,\kappa}(x^h(t))}{\partial x_i \partial x_j} &= \sum_{l,\bar{l}=1}^k \frac{c_l c_{\bar{l}}}{2\pi\kappa^2} \mathrm{e}^{\frac{-(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}} \mathrm{e}^{\frac{-(x_j^h(t)-\tilde{m}(t)-u_{\bar{l}})^2}{2\kappa^2}} f_{t,\kappa}(x^h(t)) \\
&\leq \sum_{l,\bar{l}=1}^k \frac{c_l c_{\bar{l}}}{2\pi\kappa^2} \mathrm{e}^{\frac{-(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}} \mathrm{e}^{\frac{-(x_j^h(t)-\tilde{m}(t)-u_{\bar{l}})^2}{2\kappa^2}}, \qquad (5.20)
\end{aligned}$$

where we used that $f_{t,\kappa} \leq 1$. Using this bound we get that (5.19) is bounded from above by

$$\begin{aligned}
&\mathbb{E}_n\Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{n(t)} \left|\mathbb{E}_B(x_i(t)x_j(t)) - \mathbb{E}_B(\overline{y}_i(t)\overline{y}_j(t))\right| \int_0^1 \mathbb{E}_B\Bigg(\mathbb{1}_{d(x_i^h(t), x_j^h(t))\in \bar{I}}\mathbb{1}_{x_i^h \in \mathcal{T}^\gamma_{t,\bar{I},\Sigma_h^2}} \\
&\times \sum_{l,\bar{l}=1}^k \frac{c_l c_{\bar{l}}}{2\pi\kappa^2} \mathrm{e}^{\frac{-(x_i^h(t)-\tilde{m}(t)-u_l)^2}{2\kappa^2}} \mathrm{e}^{\frac{-(x_j^h(t)-\tilde{m}(t)-u_{\bar{l}})^2}{2\kappa^2}}\Bigg) dh\Bigg). \qquad (5.21)
\end{aligned}$$

We introduce the shorthand notation

$$\begin{aligned} A_1 &= \Sigma_h^2(s)/t, \\ A_2 &= 1 - \Sigma_h^2(s)/t. \end{aligned} \tag{5.22}$$

To compute the expectation in (5.21) we fix the time of the most recent common ancestor of x_i and x_j as s and integrate over it. Then the pair $(x_i^h(t), x_j^h(t))$ has the same distribution as $(Y + X_1, Y + X_2)$, where Y, X_1, X_2 are independent centred Gaussian random variables with variance tA_1, tA_2, and tA_2, respectively. We also relax the tube condition except at the splitting time of the two particles. From this we see that the expression in (5.21) is bounded from above by

$$C \sum_{l,\bar{l}=1}^{k} c_l c_{\bar{l}} \int_{\bar{I}} |\Sigma^2(s) - \overline{\Sigma}^2(s)| \mathrm{e}^{2t-s} \tag{5.23}$$

$$\times \int_0^1 \int_{A_1\tilde{m}(t)-J(s,\gamma)}^{A_1\tilde{m}(t)+J(s,\gamma)} Q(y, u_l, t) Q(y, u_{\bar{l}}, t) \mathrm{e}^{-\frac{y^2}{2tA_1}} \frac{dy}{\sqrt{2\pi t A_1}} dh ds,$$

where $\infty > C > 0$ is a constant,

$$J(s,\gamma) = \left(\Sigma_h^2(s) \wedge (t - \Sigma_h^2(s))\right)^\gamma = ((A_1 \wedge A_2)t)^\gamma, \tag{5.24}$$

and for $1 \le l \le k$

$$Q(y, u_l, t) = \int_{-\infty}^{\infty} \mathrm{e}^{-(x+y-\tilde{m}(t)-u_l)^2/2\kappa^2} \mathrm{e}^{-\frac{x^2}{2tA_2}} \frac{dx}{\sqrt{(2\pi)^2 \kappa^2 t A_2}}. \tag{5.25}$$

We first compute $Q(y, u_l, t)$. We change variables in (5.25)

$$x = z + \frac{tA_2(\tilde{m}(t) - y - u_l)}{\kappa^2 + tA_2} \tag{5.26}$$

and obtain that (5.25) can be written as

$$\begin{aligned} Q(y, u_l, t) &= \mathrm{e}^{-\frac{(\tilde{m}(t)-y-u_l)^2}{2(\kappa^2+tA_2)}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{z^2(\kappa^2+A_2 t)}{2\kappa^2 A_2 t}}}{\sqrt{(2\pi)^2 \kappa^2 A_2 t}} dz \\ &= \frac{\mathrm{e}^{-\frac{(\tilde{m}(t)-y-u_l)^2}{2(\kappa^2+tA_2)}}}{\sqrt{2\pi(\kappa^2 + tA_2)}}. \end{aligned} \tag{5.27}$$

Plugging this into (5.24) we get

$$C \sum_{l,\bar{l}=1}^{k} c_l c_{\bar{l}} \int_{\bar{I}} |\Sigma^2(s) - \overline{\Sigma}^2(s)| \mathrm{e}^{2t-s} \tag{5.28}$$

$$\times \int_0^1 \int_{A_1\tilde{m}(t)-J(s,\gamma)}^{A_1\tilde{m}(t)+J(s,\gamma)} \frac{\mathrm{e}^{-\frac{(\tilde{m}(t)-y-u_l)^2 + (\tilde{m}(t)-y-u_{\bar{l}})^2}{2(\kappa^2+tA_2)}}}{2\pi(\kappa^2 + tA_2)} \mathrm{e}^{-\frac{y^2}{2tA_1}} \frac{dy}{\sqrt{2\pi t A_1}} dh ds,$$

In the integral with respect to y we now change variables to

$$-w = y - \frac{(2\tilde{m}(t) - u_l - u_{\bar{l}})A_1 t}{\kappa^2 + (1 + A_1)t}, \tag{5.29}$$

and drop terms that are bounded uniformly in t and κ^2 by constants to see that (5.28) is less than or equal to

$$\widetilde{C}\sum_{l,\bar{l}=1}^{k} c_l c_{\bar{l}} \int_{\bar{I}} |\Sigma^2(s)-\overline{\Sigma}^2(s)| \mathrm{e}^{2t-s} \int_0^1 \int_{\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}-J(s,\gamma)}^{\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}+J(s,\gamma)}$$

$$\times\frac{\mathrm{e}^{-\frac{\tilde{m}(t)^2}{\kappa^2+(1+A_1)t}}\mathrm{e}^{\frac{-w^2(\kappa^2+(1+A_1)t)}{2(\kappa^2+tA_2)A_1t}}}{2\pi(\kappa^2+tA_2)}\frac{dwdhds}{\sqrt{2\pi tA_1}}, \tag{5.30}$$

with $\widetilde{C}$ a new constant independent of t and κ^2. Since, for each $h\in(0,1)$,

$$\begin{aligned}&\frac{\sqrt{1+A_1}}{\sqrt{tA_1A_2}}\left(\frac{A_1A_2\tilde{m}(t)}{A_1+1}-J(s,\gamma)\right)\\ \geq\ & ((A_1\wedge A_2)\tilde{m}(t))^{-1/2}\left(\frac{1}{4}(A_1\wedge A_2)\tilde{m}(t)-(A_1\wedge A_2)^{\gamma}t^{\gamma}\right),\end{aligned} \tag{5.31}$$

which tends to $+\infty$, as $t\uparrow\infty$, we can use the Gaussian tail bound (2.11) in the integral over w to show that

$$\begin{aligned}&\mathrm{e}^{2t-s}\int_0^1 \int_{\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}-J(s,\gamma)}^{\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}+J(s,\gamma)} \frac{\mathrm{e}^{-\frac{\tilde{m}(t)^2}{\kappa^2+(1+A_1)t}}\mathrm{e}^{\frac{-w^2(\kappa^2+(1+A_1)t)}{2(\kappa^2+tA_2)A_1t}}}{2\pi(\kappa^2+tA_2)\sqrt{2\pi tA_1}}dwdh\\
\leq\ &\mathrm{e}^{2t-s}\int_0^1 \mathrm{e}^{-\frac{1}{2}\frac{\kappa^2+(1+A_1)t}{(\kappa^2+tA_2)A_1t}\left(\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}-J(s,\gamma)\right)^2}\frac{\mathrm{e}^{-\frac{\tilde{m}(t)^2}{\kappa^2+(1+A_1)t}}}{2\pi(\kappa^2+tA_2)}\\
&\times\left[\left(\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}-J(s,\gamma)\right)\frac{\kappa^2+(1+A_1)t}{(\kappa^2+tA_2)A_1t}\right]^{-1}\frac{dh}{\sqrt{2\pi tA_1}}.\\
=\ &\mathrm{e}^{2t-s}\int_0^1 \mathrm{e}^{-\frac{1}{2}\frac{\kappa^2+(1+A_1)t}{(\kappa^2+tA_2)A_1t}\left(\frac{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})}{\kappa^2+(1+A_1)t}-J(s,\gamma)\right)^2}\mathrm{e}^{-\frac{\tilde{m}(t)^2}{\kappa^2+(1+A_1)t}}\\
&\times\frac{\sqrt{A_1t}}{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})-J(s,\gamma)(\kappa^2+(1+A_1)t)}\frac{dh}{(2\pi)^{\frac{3}{2}}}.\end{aligned} \tag{5.32}$$

By the definition of $J(s,\gamma)$ we can bound (5.32) from above by

$$\widehat{C}\mathrm{e}^{2t-s}\int_0^1\frac{\sqrt{A_1t}}{A_1A_2t\tilde{m}(t)-A_1\tilde{m}(t)\kappa^2-tA_1(u_l+u_{\bar{l}})-J(s,\gamma)(\kappa^2+(1+A_1)t)}\mathrm{e}^{-\frac{(1+A_2)\tilde{m}(t)^2}{2t}+O(s^{\gamma})}\frac{dh}{(2\pi)^{\frac{3}{2}}}, \tag{5.33}$$

where $\widehat{C}<\infty$ is a constant that does not depend on t and κ^2. The denominator in the fraction appearing in the integrand equals $\sqrt{2}A_2A_1t^2(1+o(1)$, for t large, because, for all s in the integration range $\bar{I}$, it holds that $A_2t>t^{\frac{1}{3}}$ and $A_1t>t^{\frac{1}{3}}$. Using this and the fact that

$$\tilde{m}(t)^2/t=2t-\log t+O(\log(t)^2/t), \tag{5.34}$$

we see that the expression in (5.33) is smaller than

$$2\widehat{C}\int_0^1\frac{t^{1-\frac{A_1}{2}}}{A_2t\sqrt{A_1t}}\mathrm{e}^{-s+A_1t+O(s^{\gamma})}\frac{dh}{(2\pi)^{\frac{3}{2}}}, \tag{5.35}$$

Since $A_1=1-A_2$, the fraction in (5.35) is bounded by a constant times

$$\frac{t^{-1+A_2/2}}{A_2(1-A_2)^{\frac{1}{2}}}. \tag{5.36}$$

We now distinguish three regimes. If $A_2 \in (\epsilon, 1-\epsilon)$, for $\epsilon > 0$ independent of t, then the expression in (5.36) is of order $t^{-1/2}$, as $t \uparrow \infty$. If A_2 tends to 1, then for $s \in \bar{I}$,

$$\frac{t^{-1+A_2/2}}{A_2(1-A_2)^{\frac{1}{2}}} \leq t^{-1/2+1/3}, \tag{5.37}$$

which tends to zero, as $t \uparrow \infty$. Finally, when $A_2 \downarrow 0$, we get

$$\frac{t^{-1+A_2/2}}{A_2(1-A_2)^{\frac{1}{2}}} \leq t^{-1+2/3+o(1)}, \tag{5.38}$$

which tends to zero as $t \uparrow \infty$. Hence, for all $s \in \bar{I}$, (5.35) is bounded from above by

$$o(1) \int_0^1 \mathrm{e}^{-s+A_1 t+O(s^\gamma)} dh. \tag{5.39}$$

Inserting this into (5.28), and writing out $A_1 t = h\Sigma^2(s) + (1-h)\overline{\Sigma}^2(s)$, we see that

$$\begin{aligned}
&\left| \mathbb{E}_n \left(\int_0^1 (\overline{T1}) dh \right) \right| \\
&\leq o(1) \int_{\bar{I}} |\Sigma^2(s) - \overline{\Sigma}^2(s)| \int_0^1 \mathrm{e}^{-s+(h\Sigma^2(s)+(1-h)\overline{\Sigma}^2(s))+O(s^\gamma)} dh ds \\
&\leq o(1) \int_{\bar{I}} \left| \mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right| ds,
\end{aligned} \tag{5.40}$$

with $o(1)$ tending to 0, as $t \uparrow \infty$, uniformly for κ^2 small enough. This proves Lemma 4.8. □

Proof of Lemma 4.9: We split the domain of integration into three parts. First, let $\delta_3 > 0$ be such that

$$\sigma_b^2 + \tfrac{K}{2}\delta_3 < 1 \text{ and } \delta_3 < \delta_b. \tag{5.41}$$

By a Taylor expansion at zero we have

$$\Sigma^2(s) \leq (\sigma_b^2 + \tfrac{K}{2}\delta_3)s, \quad \text{for } s \in [0, \delta_3 t]. \tag{5.42}$$

Moreover, if $\delta_1 > 0$, then so is δ_0, and we then choose $\delta_3 < \delta_0^< \wedge \delta_1^<$ (with $\delta_i^< \equiv \lim_{t\uparrow\infty} \delta_i^<$); hence, for t large enough it then also holds that $\delta_3 < \delta_0^<(t) \wedge \delta_1^<(t)$.

If $\delta_1^< = 0$, we set (note that, by monotonicity, in this case $\delta_0^<(t) \wedge \delta_1^<(t) = \delta_0^<(t)$)

$$\begin{aligned}
(S1) \quad &\equiv \int_{t\delta_0^<(t)}^{\delta_3 t} \left| \mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right| ds \\
&\leq \int_{t\delta_0^<(t)}^{\delta_3 t} \left(\mathrm{e}^{-s(1-\sigma_b^2-\frac{K}{2}\delta_3)+O(s^\gamma)} + \mathrm{e}^{-s(1-\sigma_b^2-\frac{K}{2}\delta^<(t))+O(s^\gamma)} \right) ds.
\end{aligned} \tag{5.43}$$

By assumption on δ_3, $1-\sigma_b^2-\frac{K}{2}\delta_3 > 0$ and $1-\sigma_b^2-\frac{K}{2}\delta^<(t) > 0$, for all t sufficiently large. Hence

$$\lim_{t\to\infty} (S1) = 0. \tag{5.44}$$

If $\delta_1^< > 0$, we set $(S1) = 0$.

Next we choose δ_4 such that

$$\sigma_e^2 - \delta_4 \tfrac{K}{2} > 1 \text{ and } \delta_4 < \delta_e. \tag{5.45}$$

Again due to a first order Taylor expansion we have

$$\Sigma^2(t-\bar{s}) \leq t - \left(\sigma_e^2 - \tfrac{K}{2}\delta_4\right)\bar{s}, \quad \text{for } \bar{s} \in [t\delta_1^>(t), t\delta_4]. \tag{5.46}$$

Hence

$$
\begin{aligned}
(S2) \qquad &\equiv \int_{t-\delta_4 t}^{t(1-\delta_1^>(t))} \left| \mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right| ds \\
&= \int_{t\delta_1^>(t)}^{\delta_4 t} \left| \mathrm{e}^{\bar{s}-t+\Sigma^2(t-\bar{s})+O(s^\gamma)} - \mathrm{e}^{\bar{s}-t+\overline{\Sigma}^2(t-\bar{s})+O(s^\gamma)} \right| d\bar{s} \\
&\le \int_{t\delta_1^>(t)}^{\delta_4 t} \left(\mathrm{e}^{\bar{s}(1-\sigma_e^2+\frac{K}{2}\delta_4)+O(s^\gamma)} + \mathrm{e}^{\bar{s}(1-\sigma_e^2+\frac{K}{2}\delta^>(t))+O(s^\gamma)} \right) d\bar{s}. \quad (5.47)
\end{aligned}
$$

By assumption on δ_4 we have $1-\kappa_e^2+\frac{K}{2}\delta_4 < 0$ and, for t large, $1-\sigma_e^2+\frac{K}{2}\delta^>(t) < 0$. Hence

$$
\lim_{t\to\infty} (S2) = 0. \tag{5.48}
$$

We still have to control

$$
(S3) \equiv \int_{\delta_3 t}^{t-\delta_4 t} \left| \mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} - \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} \right| ds. \tag{5.49}
$$

Consider the function $A(x)$ on the interval $[\delta_3, 1-\delta_4]$. Since $A(x)$ is right-continuous, increasing and $A(x) < x$ on $(0,1)$, we know that

$$
M \equiv \inf_{x\in[\delta_3,1-\delta_4]} (x - A(x)) > 0. \tag{5.50}
$$

Then

$$
s - \Sigma^2(s) = t(s/t - A(s/t)) \ge Mt, \tag{5.51}
$$

which implies

$$
\int_{\delta_3 t}^{t-\delta_4 t} \mathrm{e}^{-s+\Sigma^2(s)+O(s^\gamma)} ds \le \mathrm{e}^{-Mt} \int_{\delta_3 t}^{t-\delta_4 t} \mathrm{e}^{O(s^\gamma)} ds, \tag{5.52}
$$

which tends to zero, as $t \uparrow \infty$. By the same argument it follows that

$$
\lim_{t\uparrow\infty} \int_{\delta_3 t}^{t-\delta_4 t} \mathrm{e}^{-s+\overline{\Sigma}^2(s)+O(s^\gamma)} ds = 0. \tag{5.53}
$$

It follows that $\lim_{t\uparrow\infty}(S3) = 0$, which concludes the proof of Lemma 4.9. □

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CHAPTER 4

Extended Convergence of the Extremal Process of Branching Brownian Motion

EXTENDED CONVERGENCE OF THE EXTREMAL PROCESS OF BRANCHING BROWNIAN MOTION

ANTON BOVIER AND LISA HARTUNG

ABSTRACT. We extend the results of Arguin et al [4] and Aïdékon et al [1] on the convergence of the extremal process of branching Brownian motion by adding an extra dimension that encodes the "location" of the particle in the underlying Galton-Watson tree. We show that the limit is a cluster point process on $\mathbb{R}_+ \times \mathbb{R}$ where each cluster is the atom of a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$ with a random intensity measure $Z(dz) \times C\mathrm{e}^{-\sqrt{2}x}$, where the random measure is explicitly constructed from the derivative martingale. This work is motivated by an analogous conjecture for the Gaussian free field by Biskup and Louidor [6].

1. INTRODUCTION

Over the last years the analysis of the extremal process of so-called *log-correlated* processes has been studied intensively. One prime example was the construction of the extremal process of branching Brownian motion [4, 1] and branching random walk [19]. The processes appearing here, Poisson point processes with random intensity (Cox processes, see [10]) decorated by a cluster process representing clusters of particles that have rather recent common ancestors, are widely believed to be universal for a wide class of log-correlated processes. In particular, it is expected for the discrete Gaussian free field, and partial results in this direction have been proven by Bramson, Ding, and Zeitouni [8] and Biskup and Louidor [6]. These results describe the statistics of the positions (= values) of the extremal points of these processes. In extreme value theory (see e.g. [18]) it is customary to give an even more complete description of extremal processes that also encode the *locations of the extreme points* ("complete Poisson convergence"). In the case of the two-dimensional Gaussian free field, Biskup and Louidor [6] conjecture[1] such a result as follows. For $(i, j) \in (1, \dots, n)^2$, let X^n be the centred Gaussian process indexed by $(1, \dots, n)^2$ with covariance[2]

$$\mathbb{E}X^n_{i,j}X^n_{k,l} = \pi G^n((i, j), (k, l)), \tag{1.1}$$

where G^n is the Green function of simple random walk on $(1, \dots, n)^2$ killed upon exiting this domain. It is conjectured that with $m_n(u) \equiv \sqrt{2}\ln n^2 - \frac{3}{2\sqrt{2}}\ln\ln n^2$, the family of

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[1]Biskup and Louidor have recently announced that they can prove this result (private communication).

[2]We change the normalisation of the variance so that the results adapt better to BBM.

point processes on $\mathbb{R}$

$$\sum_{1\leq i,j\leq n} \delta_{X_{(i,j)}-m_n} \tag{1.2}$$

converges to a process of the form

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \delta_{p_i+\Delta_j^{(i)}}, \tag{1.3}$$

where the p_i are the atoms of a Poisson point process with random intensity $Z\mathrm{e}^{-\sqrt{2}u}du$, for a random variable Z, and $\Delta^{(i)}$ are iid copies of a certain point process Δ on $[0,-\infty)$. The extended version of this conjecture reads as follows. Define the point processes

$$\mathcal{P}_n \equiv \sum_{1\leq i,j\leq n} \delta_{(i/n,j/n),X_{(i,j)}-m_n} \tag{1.4}$$

on $(0,1]^2\times\mathbb{R}$. Then $\mathcal{P}_n$ converges to a point process $\mathcal{P}$ on $(0,1]^2\times\mathbb{R}$ of the form

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \delta_{x_i,p_i+\Delta_j^{(i)}}, \tag{1.5}$$

where (x_i,p_i) are the atoms of a Poisson point process on $(0,1]^2\times\mathbb{R}$ with random intensity measure $Z(dx)\times\mathrm{e}^{-\sqrt{2}u}du$, where $Z(dx)$ is some random measure on $(0,1]^2$. In fact, Biskup and Louidor *prove* a slightly weaker result for the point process of *local extremes:* Let r_n be a sequence such that $r_n\uparrow\infty$ and $r_n/n\downarrow 0$, and define

$$\eta_n \equiv \sum_{1\leq i,j\leq n} \delta_{(i/n,j/n),\max_{(k,\ell):|k-i|<r_n,|\ell-j|} X_{(k,\ell)}-m_n}. \tag{1.6}$$

Then η_n converges to the Poisson point process on $(0,1]^2\times\mathbb{R}$ with random intensity measure $Z(dx)\times\mathrm{e}^{-\sqrt{2}u}du$,

The purpose of this article is to prove the analog of the full result for branching Brownian motion. To do so, we need to decide on what should replace the square $(0,1]^2$ in that case. Before we do this, let us briefly recall the construction of branching Brownian motion. We start with a continuous time Galton-Watson process [5] with branching mechanism $p_k, k\geq 1$, normalised such that $\sum_{i=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$ and $K=\sum_{k=1}^{\infty} k(k-1)p_k < \infty$. At any time t we may label the endpoints of the process $i_1(t),\dots,i_{n(t)}(t)$, where $n(t)$ is the number of branches at time t. Note that with this choice of normalisation, we have that $\mathbb{E}n(t)=\mathrm{e}^t$. Branching Brownian motion is then constructed by starting a Brownian motion at the origin at time zero, running it until the first time the GW process branches, and then continuing independent Brownian motions for each of the branches until their respective next branching times, and so on. We denote the positions of the $n(t)$ particles at time t by $x_1(t),\dots,x_{n(t)}(t)$. Note that, of course, the positions of these particles do not reflect the position of the particles "in the tree".

We now want to embed the leaves of a Galton-Watson process in a consistent way in some finite dimensional space (we choose $\mathbb{R}_+$) that respects the natural tree distance. Since we already know from [2] that the (normalised) genealogical distance of extreme particles is asymptotically either zero or one, one should expect that the resulting process should again be Poisson in this space. In the case of deterministic binary branching at integer times, the leaves of the tree at time n are naturally labelled by sequences $\sigma^n\equiv$

$(\sigma_1\sigma_2\dots\sigma_n)$, with $\sigma_\ell \in \{0,1\}$. These sequences can be naturally mapped into $[0,1]$ via

$$\sigma^n \mapsto \sum_{\ell=1}^{n} \sigma_\ell 2^{-\ell-1} \in [0,1]. \tag{1.7}$$

Moreover, the limit, as $n \uparrow \infty$ of the image of this map is $[0,1]$. In the next section we construct an analogous map for the Galton-Watson process.

2. The embedding

Our goal is to define a map $\gamma : \{1,\dots,n(t)\} \to \mathbb{R}_+$ in such a way that it encodes the genealogical structure of the underlying supercritical Galton-Watson process. A first step is to represent a tree by a consistent function $u : \mathbb{R}_+ \mapsto \mathbb{N}_0^{\mathbb{N}}$ of *multi-indices*. For discrete time trees, it is obvious and standard how to do this, but in continuous time this is a bit more delicate. We choose the following procedure. Denote by $W(t)$ the total number of branchings that happened in $[0,t]$. Moreover, we denote by $0 = t_0 \le t_1 \le t_2 \le \dots \le t_{W(t)}$ the branching times in increasing order. We add to the underlying Galton-Watson tree restricted to $[0,t]$, T_t, at time t_i for $i = 1,\dots,W(t)$ an extra vertex to each branch that exists at time t_i (see Figure 1. The new vertices are the thick dots). We call the resulting tree $\widetilde{T}_t$. At any of the times t_i, each vertex u of the tree will branch into $l^u(t_i)$ forward branches. Note that almost surely, at any time t_j, there will be at most one vertex for which $l^u(t_j) > 1$.

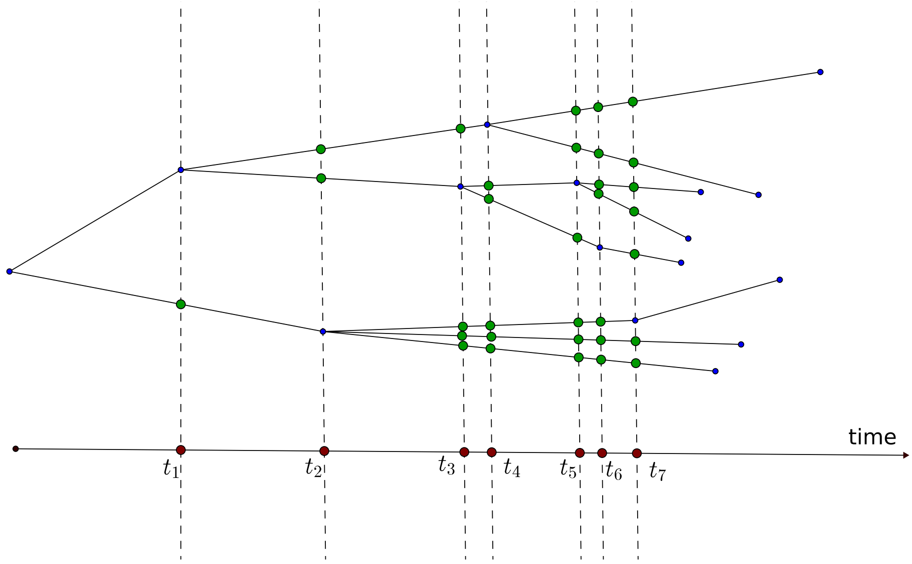

FIGURE 1. Construction of $\widetilde{T}$: The green nodes were introduced into the tree 'by hand'.

We identify $\widetilde{T}_t$ with a subset $\tau(t)$ of infinite sequences of non-negative integers. That gives us a labelling of $\widetilde{T}_t$ that is consistent in time:

- $\{(0,0,\dots)\} = u(0) = \tau(0)$.
- for all $j \ge 0$, for $t \in [t_j, t_{j+1})$, for all $u(t) \in \tau(t)$, $u(t) = u(t_j)$,
- If $u \in \tau(t_j)$ then $u + (\underbrace{0,\dots,0}_{W(t_j)\times 0}, k, 0, \dots) \in \tau(t_{j+1})$ if $0 \le k \le l^u(t_{j+1}) - 1$, where

$$l^u(t_j) = \#\{\text{ offsprings of the particle corresponding to } u \text{ at time } t_j\}. \tag{2.1}$$

We introduce for $u(t) \in \tau(t)$ the notation $u(t)|_s \in \tau(s)$ for the multi-index where the first $W(s)$ indices coincide with those of $u(t)$ and the rest are zeros. Note that this labelling is consistent in the sense that if $u(t)$ is the label of a particle at time t, then $u(t)|_s = u(s)$ is the label of the ancestor of this particle at time s. Hence we may think of $u(\infty) \in \tau(\infty)$ as a particle at "infinity" and of $(u(t), t \in \mathbb{R}_+)$ as the trajectory of a particle in the space of labels. Knowledge of all multi indices $u \in \tau(\infty)$ and of all branching times allows to reconstruct the entire infinite tree. For two particle labelled by u and v, the time of their most recent common ancestor is the then simply $d(u,v) = \sup\{t \geq 0 : u(t) = v(t)\}$.

In this way each leave of the Galton-Watson tree at time t, $i_k(t)$ with $k \in \{1, \ldots, n(t)\}$ is identified with some multi-label $u^k(t) \in \tau(t)$. Then define

$$\gamma(u(t)) \equiv \sum_{j=1}^{W(t)} u_j(t) \mathrm{e}^{-t_j}. \tag{2.2}$$

For a given u, the function $(\gamma(u(t)), t \in \mathbb{R}_+)$ describes a trajectory of a particle in $\mathbb{R}_+$. The important point is that for any given particle, this trajectory converges to some point $\gamma(u) \in \mathbb{R}_+$, as $t \uparrow \infty$, almost surely. Hence also the sets $\gamma(\tau(t))$ converge, for any realisation of the tree, to some (random) set $\gamma(\tau(\infty))$.

Remark. The labelling of the GW-tree is a slight variant of the familiar Ulam-Neveu-Harris labelling (see e.g. [14]). In our labelling the added zeros keep track of the order in which branching occurred in continuous time. We believe that this or an equivalent construction must be standard, but we have not been able to find it for continuous time trees in the literature.

In addition, in branching Brownian motion, there is also the position of the Brownian motion $x_k(t)$ of the k-th particle at time t. Hoping that there will not be too much confusion, we will often write $\gamma(x_k(t)) \equiv \gamma(u^k(t))$. Thus to any "particle" at time t we can now associate the position on $\mathbb{R} \times \mathbb{R}_+$, $(x_k(t), \gamma(u^k(t)))$.

3. The extended convergence result

In this section we state the analog to (1.5) for branching Brownian motion. First let us recall the limit of the extremal process. Bramson [9] and Lalley and Selke [17] show that

$$\lim_{t\uparrow\infty} \mathbb{P}\left(\max_{k \leq n(t)} x_k(t) - m(t) \leq x \right) = \omega(x) = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}x}} \right], \tag{3.1}$$

for some constant C and where $Z \equiv \lim_{t\uparrow\infty} Z_t$ is the limit of the derivative martingale

$$Z_t \equiv \sum_{j \leq n(t)} (\sqrt{2}t - x_j(t)) \mathrm{e}^{\sqrt{2}(x_j(t) - \sqrt{2}t)}. \tag{3.2}$$

In [4] and [1] it was shown that the process,

$$\mathcal{E}_t \equiv \sum_{k=1}^{n(t)} \delta_{x_k(t) - m(t)} \tag{3.3}$$

converges, as $t \uparrow \infty$, in law to the process

$$\mathcal{E} = \sum_{k,j} \delta_{\eta_k + \Delta_j^{(k)}}, \tag{3.4}$$

where η_k is the k-th atom of a Cox process with random intensity measure $CZ\mathrm{e}^{-\sqrt{2}y}dy$. The $\Delta_i^{(k)}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, which are copies of the limiting process

$$\Delta \overset{D}{=} \lim_{t\uparrow\infty} \sum_{i=1}^{n(t)} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{3.5}$$

where $\tilde{x}(t)$ is BBM conditioned on $\max_{j\leq n(t)} \tilde{x}_j(t) \geq \sqrt{2}t$.

Using the embedding γ defined in the previous section, we now state the following theorem, that exhibits more precisely the nature of the Poisson points and the genealogical structure of the extremal particles.

Theorem 3.1. *The point process* $\widetilde{\mathcal{E}}_t \equiv \sum_{k=1}^{n(t)} \delta_{(\gamma(u^k(t)),x_k(t)-m(t))} \to \widetilde{\mathcal{E}}$ *on* $\mathbb{R}_+ \times \mathbb{R}$*, as* $t \uparrow \infty$*, where*

$$\widetilde{\mathcal{E}} \equiv \sum_{i,j} \delta_{(q_i,p_i)+(0,\Delta_j^{(i)})}, \tag{3.6}$$

where $(q_i,p_i)_{i\in\mathbb{N}}$ *are the atoms of a Cox process on* $\mathbb{R}_+ \times \mathbb{R}$ *with intensity measure* $Z(dv) \times C\mathrm{e}^{-\sqrt{2}x}dx$*, where* $Z(dv)$ *is a random measure on* $\mathbb{R}_+$*, characterised in Lemma 3.2, and* $\Delta_j^{(i)}$ *are the atoms of independent and identically distributed point processes* $\Delta^{(i)}$ *as in (3.4) .*

Remark. The nice feature of the process $\widetilde{\mathcal{E}}_t$ is that it allows to visualise the different clusters $\Delta^{(i)}$ corresponding to the different point of the Poisson process of cluster extremes. In the process $\sum_{k=1}^{n(t)} \delta_{x_i(t)-m(t)}$ considered in earlier work, all these points get superimposed and cannot be disentangled. In other words, the process $\widetilde{\mathcal{E}}$ encodes both the values and the (rough) genealogical structure of the extremes of BBM.

The measure $Z(dv)$ in an interesting object in itself. For $v, r \in \mathbb{R}_+$ and $t > r$, we define

$$Z(v,r,t) = \sum_{j\leq n(t)} (\sqrt{2}t - x_j(t))\mathrm{e}^{\sqrt{2}(x_j(t)-\sqrt{2}t)} \mathbb{1}_{\gamma(x_i(r))\leq v}, \tag{3.7}$$

which is a truncated version of the usual derivative martingale Z_t. In particular, observe that $Z(\infty, r, t) = Z_t$.

Lemma 3.2. *For each* $v \in \mathbb{R}_+$ *the limit* $\lim_{r\uparrow\infty} \lim_{t\uparrow\infty} Z(v,r,t)$ *exists almost surely. Set*

$$Z(v) \equiv \lim_{r\uparrow\infty} \lim_{t\uparrow\infty} Z(v,r,t). \tag{3.8}$$

Then $0 \leq Z(v) \leq Z$*, where* Z *is the limit of the derivative martingale. Moreover,* $Z(v)$ *is monotone increasing in* u *and a.s. non-atomic.*

The measure $Z(v)$ is the analogue of the corresponding "derivative martingale measure" studied in Duplantier et al [11, 12] and Biskup and Louidor [6, 7] in the context of the Gaussian free field. For a review, see Rhodes and Vargas [20]. The objects are examples of what is known as *multiplicative chaos* that was introduced by Kahane [15].

4. Properties of the embedding

We need the three basic properties of γ. Lemma 4.1 states that the map $\gamma(x_k(t))$ converges for all extremal particles as $t \uparrow \infty$ and is well approximated by the information on the tree up to a fixed time r.

Lemma 4.1. *Let $D \subset \mathbb{R}$ be a compact set. Define, for $0 \leq r < t < \infty$, the events*

$$\mathcal{A}^{\gamma}_{r,t}(D) = \left\{\forall k \text{ with } x_k(t) - m(t) \in D : \gamma(x_k(t)) - \gamma(x_k(r)) \leq \mathrm{e}^{-r/2}\right\}. \tag{4.1}$$

For any $\epsilon > 0$ there exists $0 \leq r(D, \epsilon) < \infty$ such that, for any $r > r(D, \epsilon)$ and $t > 3r$

$$\mathbb{P}\left(\left(\mathcal{A}^{\gamma}_{r,t}(D)\right)^c\right) < \epsilon. \tag{4.2}$$

Proof. Set $\overline{D} \equiv \sup\{x \in D\}$ and $\underline{D} \equiv \inf\{x \in D\}$. Let $\epsilon > 0$. Then, by Theorem 2.3 of [2] there exists for each $\epsilon > 0$ $r_1 < \infty$, such that, for all $t > 3r_1$

$$\begin{aligned} \mathbb{P}\left(\left(\mathcal{A}^{\gamma}_{r,t}(D)\right)^c\right) \quad \leq \quad & \mathbb{P}\big(\exists k : x_k(t) - m(t) \in D, \forall_{s\in[r_1,t-r_1]} : x_k(s) \leq \overline{D} + E_{t,\alpha}(s) \\ & \text{but } \gamma(x_k(t)) - \gamma(x_k(r)) > \mathrm{e}^{-r/2}\big) + \epsilon/2, \end{aligned} \tag{4.3}$$

where $0 < \alpha < \frac{1}{2}$ and $E_{t,\alpha}(s) = \frac{s}{t}m(t) - f_{t,\alpha}(s)$ and $f_{t,\alpha} = (s \wedge (t-s))^{\alpha}$. Using the "many-to-one lemma" (see Theorem 8.5 of [13])), the probability in (4.3) is bounded from above by

$$\mathrm{e}^t\mathbb{P}\left(x(t) \in m(t) + D, \forall_{s\in[r_1,t-r_1]} : x(s) \leq \overline{D} + E_{t,\alpha}(s) \text{ but } \textstyle\sum_j m_j \mathrm{e}^{-\tilde{t}_j}\mathbb{1}_{\tilde{t}_j\in[r,t]} > \mathrm{e}^{-r/2}\right), \tag{4.4}$$

where x is a standard Brownian motion and $(\tilde{t}_j, j \in \mathbb{N})$ are the points of a size-biased Poisson point process with intensity measure $2dx$ independent of x, m_j are independent random variables uniformly distributed on $\{0, \dots, \tilde{l}_j - 1\}$, where finally $\tilde{l}_j$ are i.i.d. according to the size-biased offspring distribution, $\mathbb{P}(\tilde{l}_j = k) = \frac{kp_k}{2}$. Due to independence, and since $m_j \leq \tilde{l}_j$, the expression (4.4) is bounded from above by

$$\begin{aligned} & \mathrm{e}^t\mathbb{P}\left(x(t) \in m(t) + D, \forall_{s\in[r_1,t-r_1]} : x(s) \leq \overline{D} + E_{t,\alpha}(s)\right) \\ & \times\mathbb{P}\left(\textstyle\sum_j(\tilde{l}_j - 1)\mathrm{e}^{-\tilde{t}_j}\mathbb{1}_{\tilde{t}_j\in[r,t]} > \mathrm{e}^{-r/2}\right). \end{aligned} \tag{4.5}$$

The first probability in (4.5) is bounded by

$$\mathbb{P}\left(x(t) \in m(t) + D, \forall_{s\in[r_1,t-r_1]} : x(s) - \frac{s}{t}x(t) \leq \overline{D} - \underline{D} - f_{t,\alpha}(s)\right). \tag{4.6}$$

Using that $\xi(s) \equiv x(s) - \frac{s}{t}x(t)$ is a Brownian bridge from 0 to 0 in time t that is independent of $x(t)$, (4.6) equals

$$\begin{aligned} & \mathbb{P}\left(x(t) \in m(t) + D\right)\mathbb{P}\left(\forall_{s\in[r_1,t-r_1]} : \xi(s) \leq \overline{D} - \underline{D} - f_{t,\alpha}(s)\right) \\ & \leq \mathbb{P}\left(x(t) \in m(t) + D\right)\mathbb{P}\left(\forall_{s\in[r_1,t-r_1]} : \xi(s) \leq \overline{D} - \underline{D}\right). \end{aligned} \tag{4.7}$$

Using now Lemma 3.4 of [2] to bound the last factor of (4.7) we obtain that (4.7) is bounded from above by

$$\kappa\frac{r_1}{t - 2r_1}\mathbb{P}\left(x(t) \in m(t) + D\right), \tag{4.8}$$

where $\kappa < \infty$ is a positive constant. Using this as an upper bound for the first probability in (4.5) we can bound (4.5) from above by

$$\mathrm{e}^t\kappa\frac{r_1}{t - 2r_1}\mathbb{P}\left(x(t) \in m(t) + D\right)\mathbb{P}\left(\textstyle\sum_j(\tilde{l}_j - 1)\mathrm{e}^{-\tilde{t}_j}\mathbb{1}_{\tilde{t}_j\in[r,t]} > \mathrm{e}^{-r/2}\right). \tag{4.9}$$

By (5.25) of [2](resp. an easy Gaussian computation) this is bounded from above by

$$C\kappa\frac{r_1}{t - 2r_1}\mathbb{P}\left(\textstyle\sum_j(\tilde{l}_j - 1)\mathrm{e}^{-\tilde{t}_j}\mathbb{1}_{\tilde{t}_j\in[r,t]} > \mathrm{e}^{-r/2}\right), \tag{4.10}$$

for some positive constant $C < \infty$. Using the Markov inequality, (4.10) is bounded from above by

$$C\kappa \frac{tr_1}{t-2r_1} \mathrm{e}^{r/2} \mathbb{E}\left(\sum_j (\tilde{l}_j - 1) \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [r,t]} \right), \tag{4.11}$$

We condition on the σ-algebra $\mathcal{F}$ generated by the Poisson points. Using that $\tilde{l}_j$ is independent of the Poisson point process $(\tilde{t}_j)_j$ and $\sum_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [r,t]}$ is measurable with respect to $\mathcal{F}$ we obtain that (4.11) is equal to

$$\begin{aligned} & C\kappa \frac{tr_1}{t-2r_1} \mathrm{e}^{r/2} \mathbb{E}\left(\mathbb{E}\left(\sum_j (\tilde{l}_j - 1) \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [r,t]} | \mathcal{F} \right) \right) \\ & = C\kappa \frac{tr_1}{t-2r_1} \mathrm{e}^{r/2} \mathbb{E}\left(\sum_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [r,t]} \mathbb{E}\left((\tilde{l}_j - 1) | \mathcal{F} \right) \right). \end{aligned} \tag{4.12}$$

Since $\mathbb{E}(l_j - 1) = \sum_k \frac{1}{2}(k-1)k = K/2 < \infty$ we have that (4.12) is equal to

$$C\kappa K/2 \frac{tr_1}{t-2r_1} \mathrm{e}^{r/2} \mathbb{E}\left(\sum_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [r,t]} \right). \tag{4.13}$$

By Campbell's theorem (see e.g [16]), (4.13) is equal to

$$C\kappa K/2 \frac{tr_1}{t-2r_1} \mathrm{e}^{r/2} \int_r^t \mathrm{e}^{-x} 2dx \le C\kappa K \frac{tr_1}{t-2r_1} \mathrm{e}^{-r/2}, \tag{4.14}$$

which can be made smaller than $\epsilon/2$ for all r sufficiently large and $t > 3r$.

$\square$

The second lemma now ensures that γ maps particles are with a low probability to a very small neighbourhood of a fixed $a \in \mathbb{R}$.

Lemma 4.2. *Let $a \in \mathbb{R}_+$ and $D \subset \mathbb{R}$ be a compact set. Define the event*

$$\mathcal{B}^{\gamma}_{r,t}(D, a, \delta) = \{\forall k \text{ with } x_k(t) - m(t) \in D \colon \gamma(x_k(r)) \notin [a-\delta, a]\} \tag{4.15}$$

For any $\epsilon > 0$ there exists $\delta > 0$ and $r(a, D, \delta, \epsilon)$ such that for any $r > r(a, D, \delta, \epsilon)$ and $t > 3r$

$$\mathbb{P}\left(\left(\mathcal{B}^{\gamma}_{r,t}(D, a, \delta) \right)^c \right) < \epsilon. \tag{4.16}$$

Proof. Following the proof of Lemma 4.1 step by step we arrive at the bound

$$\mathbb{P}\left(\left(\mathcal{B}^{\gamma}_{r,t}(D, a, \delta) \right)^c \right) \le C\kappa \frac{tr_1}{t-2r_1} \mathbb{P}\left(\sum_j m_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [0,r]} \in [a-\delta, a] \right). \tag{4.17}$$

We rewrite the probability in (4.17) in the form

$$\sum_{i^*=1}^{\infty} \mathbb{P}\left(i^* = \inf\{i : m_i \neq 0\}, \sum_{j \ge i^*} m_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [0,r]} \in [a-\delta, a] \right). \tag{4.18}$$

Consider first $\mathbb{P}\left(i^* = \inf\{i : m_i \neq 0\}\right)$. This probability is equal to

$$\mathbb{P}\left(\forall_{i \le i^*} : m_i = 0 \text{ and } m_{i^*} \neq 0 \right) = \mathbb{E}\left[\left(1 - \frac{1}{l_{i^*}} \right) \prod_{j=1}^{i^*-1} \frac{1}{l_j} \right]. \tag{4.19}$$

Using that the l_j are iid together with the simple bound $\mathbb{E}\left(l_j^{-1}\right) \le \frac{1+p_1}{2}$, we see that (4.19) is bounded from above by

$$\left(\frac{1+p_1}{2} \right)^{i^*-1}. \tag{4.20}$$

Since $\frac{1+p_1}{2} < 1$ by assumption on p_1 we can choose for each $\epsilon' > 0$ $K(\epsilon') < \infty$ such that

$$\sum_{i^*=K(\epsilon')+1}^{\infty} \left(\frac{1+p_1}{2}\right)^{i^*-1} < \epsilon'. \tag{4.21}$$

Hence we bound (4.18) by

$$\sum_{i^*=1}^{K(\epsilon')} \mathbb{P}\left(i^* = \inf\{i : m_i \neq 0\}, \sum_{j\geq i^*} m_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j \in [0,r]} \in [a-\delta, a]\right) + \epsilon'. \tag{4.22}$$

We rewrite

$$\sum_{j\geq i^*} m_j \mathrm{e}^{-\tilde{t}_j} \mathbb{1}_{\tilde{t}_j\in[0,r]} = m_{i^*} \mathrm{e}^{-\tilde{t}_{i^*}} \mathbb{1}_{\tilde{t}_{i^*}\in[0,r]} \left(1 + m_{i^*}^{-1} \sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [0, r - t_{i^*}]}\right) \tag{4.23}$$

Next, we estimate the probability that $\tilde{t}_{i^*}$ is large. Observe that $\tilde{t}_{i^*} = \sum_{i=1}^{i^*} s_i$ where s_i are iid exponentially distributed random variables with parameter 2. This implies that $\tilde{t}_{i^*}$ is Erlang$(2, i^*)$. Thus

$$\mathbb{P}\left(\tilde{t}_{i^*} > r^\alpha\right) = \mathrm{e}^{-2r^\alpha} \sum_{i=0}^{i^*} \frac{(2r^\alpha)^i}{i!} \leq \tilde{C}(K(\epsilon')) b (2r^\alpha)^{K(\epsilon')} \mathrm{e}^{-2r^\alpha}, \text{ for all } i^* \leq K(\epsilon). \tag{4.24}$$

Next we want to replace $\tilde{t}_{i^*}$ in the indicator function in (4.23) by a non-random quantity r^α, for some $0 < \alpha < 1$, in order to have a bound that depends only on the differences $\tilde{t}_j - \tilde{t}_{i^*}$. Note first that

$$\begin{aligned} &\sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [0, r - t_{i^*}]} - \sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [0, r - r^\alpha]} \\ &= \sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [r - r^\alpha, r - t_{i^*}]} \leq \sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [r - r^\alpha, r]}. \end{aligned} \tag{4.25}$$

Using the fact that $m_j \leq \tilde{l}_j - 1$ for all j and the Markov inequality, we get that

$$\begin{aligned} &\mathbb{P}\left(\sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - \tilde{t}_{i^*} \in [r - r^\alpha, r]} > \mathrm{e}^{-r/2}\right) \\ &\leq \mathrm{e}^{r/2} \mathbb{E}\left(\sum_{j>i^*} (\tilde{l}_j - 1) \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - \tilde{t}_{i^*} \in [r - r^\alpha, r]}\right). \end{aligned} \tag{4.26}$$

Using Campbell's theorem as in (4.12), we see that the second line in (4.26) is equal to

$$\mathrm{e}^{r/2} K/2 \int_{r-r^\alpha}^{r} \mathrm{e}^{-x} 2dx = K\left(\mathrm{e}^{-r/2 + r^\alpha} - \mathrm{e}^{-r/2}\right). \tag{4.27}$$

For any $\epsilon' > 0$, there exists $r_0 < \infty$, such that for all $r > r_0$, the probabilities in (4.24) and (4.26) are smaller than ϵ'. On the the event

$$\mathcal{D} = \{t_{i^*} \leq r^\alpha\} \cap \left\{\sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_i^* \in [r - r^\alpha, r]} \leq \mathrm{e}^{-r/2}\right\}, \tag{4.28}$$

which has probability at least $1 - 2\epsilon'$, we can bound (4.22) in a nice way. Namely, since $m_{i^*} \geq 1$ by definition and m_j are chosen uniformly from $(0, \ldots, l_j - 1)$ and independent of $\{t_j\}_{j\geq 1}$. Moreover, $\sum_{j>i^*} m_j \mathrm{e}^{-(\tilde{t}_j - \tilde{t}_{i^*})} \mathbb{1}_{\tilde{t}_j - t_{i^*} \in [0, r - r^\alpha]} \geq 0$ is also independent of t_{i*}. It follows that (4.22) is bounded from above by

$$\sum_{i^*=1}^{K(\epsilon')} \mathbb{P}\left(i^* = \inf\{i : m_i \neq 0\}\right) \max_{b\in[0,1]} \mathbb{P}\left(\{\mathrm{e}^{-\tilde{t}_{i^*}} \in [b-\delta-\mathrm{e}^{-r/2}, b]\} \wedge \{t_{i^*} \leq r^\alpha\}\right) + 3\epsilon'. \tag{4.29}$$

Using the bound on the first probability in (4.29) given in (4.20), one sees that (4.29) is bounded from above by

$$\sum_{i^*=1}^{K(\epsilon')} \left(\frac{1+p_1}{2}\right)^{i^*-1} \max_{b\in[\delta+\mathrm{e}^{-r^\alpha}+\mathrm{e}^{-r/2},1]} \mathbb{P}\left(t_{i^*} \in \left[-\log b, -\log\left(b-\delta-\mathrm{e}^{-r/2}\right)\right]\right) + 3\epsilon' \tag{4.30}$$

Recalling that t_{i^*} is Erlang$(2, i^*)$ distributed, we have that

$$\begin{aligned} &\mathbb{P}\left(t_{i^*} \in \left[-\log b, -\log\left(b-\delta-\mathrm{e}^{-r/2}\right)\right]\right) \\ &= \sum_{i=0}^{i^*-1} \frac{1}{i!}\left(f(b) - f(b-\delta-\mathrm{e}^{-r/2})\right), \end{aligned} \tag{4.31}$$

where we have set $f(x) = 2x\left(-2\log(x)\right)^i$. By the mean value theorem, uniformly on $b \in [\delta + \mathrm{e}^{-r^\alpha} + \mathrm{e}^{-r/2}, 1]$,

$$\left(f(b) - f(b-\delta-\mathrm{e}^{-r/2})\right) \leq 2(-2\log(\delta))^{i^*}(i^*+1)\left(\delta+\mathrm{e}^{-r/2}\right). \tag{4.32}$$

Inserting this bound into (4.31), we get that, for $i^* \leq K(\epsilon')$,

$$\max_{b\in[\delta+\mathrm{e}^{-r^\alpha}+\mathrm{e}^{-r/2},1]} \mathbb{P}\left(t_{i^*} \in \left[-\log b, -\log\left(b-\delta-\mathrm{e}^{-r/2}\right)\right]\right) \tag{4.33}$$

$$\begin{aligned} &\leq 2\sum_{i=1}^{i^*-1} \frac{1}{(i-1)!}(-2\log(\delta))^{i-1}(\delta+\mathrm{e}^{-r/2}) \\ &\leq C(K(\epsilon'))\left(-\log(\delta)\right)^{K(\epsilon')}(\delta+\mathrm{e}^{-r/2}), \end{aligned} \tag{4.34}$$

for some constant $C(K(\epsilon')) < \infty$. Now the right-hand side of (4.34) can be made smaller than ϵ' by choosing r large enough and δ small enough. Collecting the bounds in (4.24), (4.26) and (4.34) implies (4.16) if $\epsilon' = \epsilon/4$ □

The following lemma asserts that any two points that get close to the maximum of BBM with have distinct images under the map γ, unless their genealogical distance is large.

Lemma 4.3. *Let $D \subset \mathbb{R}$ be a compact set. For any $\epsilon > 0$ there exists $\delta > 0$ and $r(\delta, \epsilon)$ such that for any $r > r(\delta, \epsilon)$ and $t > 3r$*

$$\mathbb{P}\left(\exists_{i,j\leq n(t):d(x_i(t),x_j(t))\leq r} : x_i(t), x_j(t) \in m(t) + D, |\gamma(x_i(t)) - \gamma(x_j(t))| \leq \delta\right) < \epsilon. \tag{4.35}$$

Proof. To control (4.35), we first use that, by Theorem 2.1 in [2], for any ϵ', there is $r_1 < \infty$, such that, for all $t \geq 3r_1$, and $r \leq t/3$, the event

$$\{\exists_{i,j\leq n(t):d(x_i(t),x_j(t))\in(r_1,r)}, x_i(t), x_j(t) \in m(t) + D\} \tag{4.36}$$

has probability smaller than ϵ'. Therefore,

$$\begin{aligned} &\mathbb{P}\left(\exists_{i,j\leq n(t):d(x_i(t),x_j(t))\leq r}, x_i(t), x_j(t) \in m(t) + D, |\gamma(x_i(t)) - \gamma(x_j(t))| \leq \delta\right) \quad (4.37) \\ &\leq \mathbb{P}\left(\exists_{i,j\leq n(t):d(x_i(t),x_j(t))\leq r_1} : x_i(t), x_j(t) \in m(t) + D, |\gamma(x_i(t)) - \gamma(x_j(t))| \leq \delta\right) + \epsilon'. \end{aligned}$$

The nice feature of the probability in the last line is that r_1 is now independent of r. At the expense of one more ϵ', we can introduce in addition the condition that the paths on $x_i(t), x_j(t)$ are localised in $E_{t,\alpha}$ over the interval $[r_2, t-r_2]$, for some $r_1 < r_2 < \infty$, independent of t. Then a second moment estimate (also known as the many-to-two lemma), shows that

$$\begin{aligned}
&\mathbb{P}\left(\exists_{i,j\leq n(t):d(x_i(t),x_j(t))\leq r_1} : x_i(t), x_j(t) \in m(t)+D, |\gamma(x_i(t)) - \gamma(x_j(t))| \leq \delta\right) \\
&\leq \mathrm{e}^{2r_1} K \mathbb{P}\Big(\exists_{i\leq n^{(1)}(t-r_1), j\leq n^{(2)}(t-r_1)} \tilde{x}_i^{(1)}(t-r_1), \tilde{x}_j^{(2)}(t-r_1) \in m(t)+\widetilde{D}, \forall_{s\in[r_2,t-r_2]}, \\
&\qquad \tilde{x}_i^{(1)}(s), x_j^{(2)} \leq \overline{D} + E_{t,\alpha}(s), k=1,2, |\gamma(x_i^{(1)}(t)) - \gamma(x_j^{(2)}(t))| \leq \delta\Big) + \epsilon',
\end{aligned} \tag{4.38}$$

where we write $x_i^{(k)}(t) = x_k(r_1) + \tilde{x}^{(k)}(t-r_1)$ and $\widetilde{D}$ is a finite enlargement of D such that $D + x_k(r_1) \subset \widetilde{D}$ with probability at least $1-\epsilon'$, and $\overline{D}$ is the supremum of $\widetilde{D}$. Using independence of the branches $\widetilde{x}^{(k)}$ and the same arguments as in Lemma 4.1, we see that the probability in the last line is bounded from above by

$$\left(C\kappa \frac{tr_2}{t-2r_2}\right)^2 \mathbb{P}\left(\left|\gamma(x_1(r_1)) - \gamma(x_2(r_1)) + \textstyle\sum_k m_k^j \mathrm{e}^{-\tilde{t}_k^j} - \sum_{k'} m_{k'}^i \mathrm{e}^{-\tilde{t}_{k'}^i}\right| \leq \delta\right), \tag{4.39}$$

where $(\tilde{t}_k^j, k \in \mathbb{N})$ and $(\tilde{t}_{k'}^i, k' \in \mathbb{N})$ are the points of independent Poisson point processes with intensity $2dx$ restricted to $[r_1, t]$. Moreover, $l_k^j, l_{k'}^i$ are i.i.d. according to the size-biased offspring distribution and m_k^j resp. $m_{k'}^i$ are uniformly distributed on $\{0, \ldots, l_k^j - 1\}$ resp. $\{0, \ldots, l_{k'}^i - 1\}$. We rewrite (4.39) as

$$\mathbb{P}\left(\textstyle\sum_k m_k^j \mathrm{e}^{-\tilde{t}_k^j} \mathbb{1}_{\tilde{t}_k^j\in[r_1,t]} \in \gamma(x_2(r_1)) - \gamma(x_1(r_1)) + \sum_{k'} m_{k'}^i \mathrm{e}^{-\tilde{t}_{k'}^i} \mathbb{1}_{\tilde{t}_{k'}^j\in[r_1,t]} + [-\delta,\delta]\right). \tag{4.40}$$

As in (4.18) we rewrite the probability in (4.40) as

$$\begin{aligned}
&\sum_{l=1}^{\infty} \mathbb{P}\Big(l = \inf\{k : m_k^j \neq 0\}, \\
&\qquad \sum_{k\geq l} m_k^j \mathrm{e}^{-\tilde{t}_k^j} \in \gamma(x_1(r_1)) - \gamma(x_2(r_1)) + \sum_{k'} m_{k'}^i \mathrm{e}^{-\tilde{t}_{k'}^i} + [-\delta,\delta]\Big).
\end{aligned} \tag{4.41}$$

Due to the independence of $(\tilde{t}_k^j, k \in \mathbb{N})$ and $(\tilde{t}_{k'}^i, k' \in \mathbb{N})$ we can proceed as with (4.18) in the proof of Lemma 4.2 to make (4.41) as small as desired by choosing δ small enough. The prefactor in (4.39) tends to a constant as $t \uparrow \infty$, and the additional prefactor from (4.38) is independent of t and δ. This implies the assertion of Lemma 4.3. □

5. THE q-THINNING

The proof of the convergence of $\sum_{i=1}^{n(t)} \delta_{(\gamma(x_i(t)), x_i(t)-m(t))}$ comes in two main steps. In a first step, we show that the points of the local extrema converge to the desired Poisson point process. To make this precise, we work with the concept of thinning classes that was already introduced in [3]. We repeat the construction here for completeness and introduce the corresponding notation.

Assume here and in the sequel that the particles at time t are labeled in decreasing order

$$x_1(t) \geq x_2(t) \geq \cdots \geq x_{n(t)}(t), \tag{5.1}$$

and set $\bar{x}_k(t) \equiv x_k(t) - m(t)$. Let

$$\bar{Q}(t) = \{\bar{Q}_{i,j}(t)\}_{i,j \le n(t)} \equiv \{t^{-1} Q_{i,j}(t)\}_{i,j \le n(t)}, \tag{5.2}$$

where

$$Q_{i,j} = \sup\{s \le t : x_i(s) = x_j(s)\}. \tag{5.3}$$

$(\mathcal{E}(t), \bar{Q}(t))$ admits the following thinning. For any $q \ge 0$ the following is true: If $\bar{Q}_{i,j}(t) \ge q \wedge \bar{Q}_{j,k}(t) \ge q$, then $\bar{Q}_{i,k}(t) \ge q$. Therefore, the sets $\{i, j \in \{1, \ldots, n(t)\} : \bar{Q}_{i,j}(t) \ge q$ form a partition of the set $\{1, \ldots, n(t)\}$ into equivalence classes. We select the maximal particle of each equivalence class as representative in the following recursive manner:

$$\begin{aligned} &i_1 = 1 \\ &i_k = \min\{j \ge i_{k-1} : \bar{Q}_{i,j}(t) \le q, \; \forall i \le k-1\}, \end{aligned} \tag{5.4}$$

if such an j exists. If no such j exists, we denote $k-1 = n^*(t)$ and terminate the procedure. The q- thinning process of $(\mathcal{E}(t), \bar{Q}(t))$, denoted by $\mathcal{E}^{(q)}(t)$ is defined by

$$\mathcal{E}^{(q)}(t) = \sum_{k=1}^{n^*(t)} \delta_{\bar{x}_{i_k}(t)}. \tag{5.5}$$

6. Extended convergence of thinned point process

For $r_d \in \mathbb{R}_+$ and $t > 3r_d$ consider the thinned process $\mathbb{E}\mathrm{e}^{(r_d/t)}(t)$. Observe that, for $R_t = m(t) - m(t - r_d) - \sqrt{2} r_d = o(1)$, we have

$$\mathcal{E}^{(r_d/t)}(t) \stackrel{D}{\equiv} \sum_{j=1}^{n(r_d)} \delta_{x_j(r_d) - \sqrt{2} r_d + M_j(t-r_d) + R_t} \tag{6.1}$$

where $M_j(t - r_d) \equiv \max_{k \le n_j(t-r_d)} x_k^{(j)}(t - r_d) - m(t - r_d)$ and $x^{(j)}$ independent BBM's (see (3.15) in [3]). Then

Proposition 6.1. *Let $\mathcal{E}^{(r_d/t)}(t)$ be defined in (5.5). Then*

$$\lim_{r_d \uparrow \infty} \lim_{t \uparrow \infty} \sum_{k=1}^{n^*(t)} \delta_{(\gamma(x_{i_k}(t)), \bar{x}_{i_k}(t)} \stackrel{D}{=} \sum_i \delta_{(q_i, p_i)} \equiv \widehat{\mathcal{E}}, \tag{6.2}$$

where $(q_i, p_i)_{i \in \mathbb{N}}$ are the points of the Cox process $\widehat{\mathcal{E}}$ with intensity measure $Z(dv) \times C\mathrm{e}^{-\sqrt{2}x} dx$ with the random measure $Z(dv)$ defined in (3.8). Moreover,

$$\lim_{r \uparrow \infty} \lim_{r_d \uparrow \infty} \sum_{j=1}^{n(r_d)} \delta_{(\gamma(x_j(r)), x_j(r_d) - \sqrt{2} r_d + M_j)} \stackrel{D}{=} \widehat{\mathcal{E}}, \tag{6.3}$$

where M_j are i.i.d with law ω defined in (3.1).

The proof of Proposition 6.1 relies in Lemma 3.2 which we now prove.

Proof of Lemma 3.2. For $v, r \in \mathbb{R}_+$ fixed, the process $Z(v, r, t)$ defined in (3.7) is a martingale in $t > r$ (since $Z(\infty, r, t)$ is the derivative martingale and $\mathbb{1}_{\gamma(x_i(r)) \le v}$ does not depend on t). To see that $Z(v,r,t)$ converges a.s. as $t\uparrow\infty$, note that

$$\begin{aligned} Z(v,r,t) &= \sum_{i=1}^{n(r)} \mathbb{1}_{\gamma(x_i(r))\le v} \mathrm{e}^{\sqrt{2}(x_i(r)-\sqrt{2}r)} \left(\sum_{j=1}^{n^{(i)}(t-r)} \left(\sqrt{2}r - x_i(r)\right) \mathrm{e}^{\sqrt{2}(x_j^{(i)}(t-r)-\sqrt{2}(t-r))} \right. \\ &\quad \left. + \sum_{j=1}^{n^{(i)}(t-r)} \left(\sqrt{2}(t-r) - x_j^{(i)}(t-r)\right) \mathrm{e}^{\sqrt{2}(x_j^{(i)}(t-r)-\sqrt{2}(t-r))} \right) \\ &= \sum_{i=1}^{n(r)} \mathbb{1}_{\gamma(x_i(r))\le v} \mathrm{e}^{\sqrt{2}(x_i(r)-\sqrt{2}r)} \left(\sqrt{2}r - x_i(r)\right) Y_{t-r}^{(i)} \\ &\quad + \sum_{i=1}^{n(r)} \mathbb{1}_{\gamma(x_i(r))\le v} \mathrm{e}^{\sqrt{2}(x_i(r)-\sqrt{2}r)} Z_{t-r}^{(i)}. \end{aligned} \tag{6.4}$$

Here $Z_t^{(i)}, i\in\mathbb{N}$ are iid copies of the derivative martingale, and $Y_t^{(i)}$ are iid copies of the McKean martingale. Lalley and Sellke proved in [17] that $\lim_{t\uparrow\infty} Y_t = 0$, a.s. while $\lim_{t\uparrow\infty} Z_t = Z$ is a non-trivial random variable. This implies that

$$\lim_{t\uparrow\infty} Z(v,r,t) \equiv Z(v,r) = \sum_{i=1}^{n(r)} \mathrm{e}^{\sqrt{2}(x_i(r)-\sqrt{2}r)} Z^{(i)} \mathbb{1}_{\gamma(x_i(r))\le v}, \tag{6.5}$$

where $Z^{(i)}$, $i\in\mathbb{N}$ are iid copies of Z. To show that $Z(v,r)$ converges, as $r\uparrow\infty$, we go back to (3.7). Note that for fixed v, $\mathbb{1}_{\gamma(x_i(r))\le v}$ is monotone decreasing in r. On the other hand, Lalley and Sellke have shown that $\min_{i\le n(t)}\left(\sqrt{2}t - x_i(t)\right) \to +\infty$, almost surely, as $t\uparrow\infty$. Therefore, the part of the sum in (3.7) that involves negative terms (namely those for which $x_i(t) > \sqrt{2}t$) converges to zero, almost surely. The remaining part of the sum is decreasing in r, and this implies that the limit, as $t\uparrow\infty$, is monotone decreasing almost surely. Moreover, $0\le Z(v,r)\le Z$, a.s., where Z is the almost sure limit of the derivative martingale. Thus $\lim_{r\uparrow\infty} Z(v,r)\equiv Z(v)$ exists. Finally, $0\le Z(v)\le Z$ and $Z(v)$ is an increasing function of v because $Z(v,r)$ is increasing in v, a.s., for each r.

To show that $Z(du)$ is nonatomic, fix $\epsilon,\delta>0$ and let $D\subset\mathbb{R}$ be compact. By Lemma 4.3 there exists $r_1(\epsilon,\delta)$ such that, for all $r>r_1(\epsilon,\delta)$ and $t>3r$,

$$\mathbb{P}\left(\exists_{i,j\le n(t)} : d(x_i(t),x_j(t))\le r, x_i(t),x_j(t)\in m(t)+D, |\gamma(x_i(t))-\gamma(x_j(t))|\le\delta\right) < \epsilon. \tag{6.6}$$

Rewriting (6.6) in terms of the thinned process $\mathcal{E}^{(r/t)}(t)$ gives

$$\mathbb{P}\left(\exists_{i_k,i_{k'}} : \bar{x}_{i_k},\bar{x}_{i_{k'}}\in m(t)+D, |\gamma(\bar{x}_{i_k}(t))-\gamma(\bar{x}_{i_{k'}}(t))|\le\delta\right)\le\epsilon. \tag{6.7}$$

Assuming for the moment that $\mathcal{E}^{(r/t)}(t)$ converges as claimed in Proposition 6.1, this implies that for any $\epsilon>0$, for small enough $\delta>0$,

$$\mathbb{P}\left(\exists\delta>0 : \exists i\ne j : |q_i-q_j|<\delta\right)<\epsilon. \tag{6.8}$$

This could not be true if $Z(du)$ had an atom. This proves Lemma 3.2 provided we can show convergence of $\mathcal{E}^{(r/t)}(t)$. $\square$

The proof of Proposition 6.1 uses the properties of the map γ obtained in Lemma 4.1 and 4.2. In particular, we use that, in the limit as $t\uparrow\infty$, the image of the extremal particles under γ converges and that essentially no particle is mapped too close to the boundary of any given compact set. Having these properties at hand we can use the same procedure as in the proof of Proposition 5 in [3]. Finally, we use Lemma 3.2 to deduce Proposition 6.1.

Proof of Proposition 6.1. We show the convergence of the Laplace functionals. Let $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be a measurable function with compact support. For simplicity we start by looking at simple functions of the form

$$\phi(x,y) = \sum_{i=1}^{N} a_i \mathbb{1}_{A_i \times B_i}(x,y), \tag{6.9}$$

where $A_i = [\underline{A}_i, \overline{A}_i]$ and $B_i = [\underline{B}_i, \overline{B}_i]$ for $N \in \mathbb{N}$, $i = 1, \dots, N$, $a_i, \underline{A}_i, \overline{A}_i \in \mathbb{R}_+$, and $\underline{B}_i, \overline{B}_i \in \mathbb{R}$. The extension to general functions ϕ then follows by monotone convergence. For such ϕ, we consider the Laplace functional

$$\Psi_t(\phi) \equiv \mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)} \phi\left(\gamma(x_{i_k}(t)), \bar{x}_{i_k}(t)\right)\right)\right]. \tag{6.10}$$

The idea is that the function γ only depends on the early branchings of the particle. To this end we insert the identity

$$1 = \mathbb{1}_{\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi)} + \mathbb{1}_{\left(\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi)\right)^c} \tag{6.11}$$

into (6.10), where $\mathcal{A}^{\gamma}_{r,t}$ is defined in (4.1), and by $\mathrm{supp}_y \phi$ we mean the support of ϕ with respect to the second varible. By Lemma 4.1 we have that, for all $\epsilon > 0$, there exists r_ϵ such that, for all $r > r_\epsilon$,

$$\mathbb{P}\left(\left(\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi)\right)^c\right) < \epsilon, \tag{6.12}$$

uniformly in $t > 3r$. Hence it suffices to show the convergence of

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)} \phi\left(\gamma(x_{i_k}(t)), \bar{x}_{i_k}(t)\right)\right) \mathbb{1}_{\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi)}\right]. \tag{6.13}$$

We introduce yet another identity into (6.13), namely

$$1 = \mathbb{1}_{\bigcap_{i=1}^{N}\left(\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \underline{A}_i) \cap \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i)\right)} + \mathbb{1}_{\left(\bigcap_{i=1}^{N}\left(\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \underline{A}_i) \cap \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i)\right)\right)^c}, \tag{6.14}$$

where we use the shorthand notation $\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i) \equiv \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i, \mathrm{e}^{-r/2})$ (recall (4.15)). By Lemma 4.2 there exists for all $\epsilon > 0$ $\bar{r}_\epsilon$ such that for all $r > \bar{r}_\epsilon$ and uniformly in $t > 3r$

$$\mathbb{P}\left(\left(\cap_{i=1}^{N}\left(\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \underline{A}_i) \cap \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i)\right)\right)^c\right) < \epsilon. \tag{6.15}$$

Hence we only have to show the convergence of

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)} \phi\left(\gamma(x_{i_k}(t)), \bar{x}_{i_k}(t)\right)\right) \mathbb{1}_{\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi) \cap \left(\bigcap_{i=1}^{N}\left(\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \underline{A}_i) \cap \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i)\right)\right)}\right]. \tag{6.16}$$

Observe that on the event in the indicator function in the the last line the following holds: If for any $i \in \{1, \dots, N\}$, $\gamma(x_k(t)) \in [\underline{A}_i, \overline{A}_i]$ and $\bar{x}_k(t) \in \mathrm{supp}_y \phi$ then also $\gamma(x_k(r)) \in [\underline{A}_i, \overline{A}_i]$, and vice versa. Hence (6.16) is equal to

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)} \phi\left(\gamma(x_{i_k}(r)), \bar{x}_{i_k}(t)\right)\right) \mathbb{1}_{\mathcal{A}^{\gamma}_{r,t}(\mathrm{supp}_y \phi) \cap \left(\bigcap_{i=1}^{N}\left(\mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \underline{A}_i) \cap \mathcal{B}^{\gamma}_{r,t}(\mathrm{supp}_y \phi, \overline{A}_i)\right)\right)}\right]. \tag{6.17}$$

Now we apply again Lemma 4.1 and Lemma 4.2 to see that the quantity in (6.17) is equal to

$$\mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)}\phi\left(\gamma(x_{i_k}(r)),\bar{x}_{i_k}(t)\right)\right)\right]+O(\epsilon). \tag{6.18}$$

Introducing a conditional expectation given $\mathcal{F}_{r_d}$, we get (analogous to (3.16) in [3]) as $t\uparrow\infty$ that (6.18) is equal to

$$\begin{aligned}
&\lim_{t\uparrow\infty}\mathbb{E}\left[\exp\left(-\sum_{k=1}^{n^*(t)}\phi\left(\gamma(x_{i_k}(r)),\bar{x}_{i_k}(t)\right)\right)\right] \qquad (6.19)\\
&=\lim_{t\uparrow\infty}\mathbb{E}\left[\prod_{j=1}^{n(r_d)}\mathbb{E}\left[\mathrm{e}^{-\phi(\gamma(x_j(r)),x_j(r_d)-m(t)+m(t-r_d)+\max_{i\leq n^{(j)}(t-r_d)}x_i^{(j)}(t-r_d)-m(t-r_d))}\middle|\mathcal{F}_{r_d}\right]\right]\\
&=\mathbb{E}\left[\prod_{j=1}^{n(r_d)}\mathbb{E}\left[\mathrm{e}^{-\phi(\gamma(x_j(r)),x_j(r_d)-\sqrt{2}r_d+M)}\middle|\mathcal{F}_{r_d}\right]\right],
\end{aligned}$$

where M is the limit of the rescaled maximum of BBM whose distribution is given in (3.1). The last expression is completely analogous to Eq. (3.17) in [3]. Following the analysis of this expression up to Eq. (3.25) in [3], we find that (6.19) is equal to

$$c_{r_d}\mathbb{E}\Big[\exp\Big(-C\sum_{j\leq n(r_d)}y_j(r_d)\mathrm{e}^{-\sqrt{2}y_j(r_d)}\sum_{i=1}^{N}(1-\mathrm{e}^{a_i})\mathbb{1}_{A_i}(\gamma(x_j(r)))\big(\mathrm{e}^{-\sqrt{2}\,\underline{B}_i}-\mathrm{e}^{-\sqrt{2}\,\overline{B}_i}\big)\Big)\Big], \tag{6.20}$$

where $y_j(r_d)=x_j(r_d)-\sqrt{2}r_d$ and $\lim_{r_d\uparrow\infty}c_{r_d}=1$, and C is the constant from (3.1). Using Lemma 3.2 (6.20) is in the limit as $r_d\uparrow\infty$ and $r\uparrow\infty$ equal to

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-C\sum_{i=1}^{N}(1-\mathrm{e}^{a_i})\big(\mathrm{e}^{-\sqrt{2}\underline{B}_i}-\mathrm{e}^{-\sqrt{2}\,\overline{B}_i}\big)\right)\left(Z(\overline{A_i})-Z(\underline{A_i})\right)\right] \qquad (6.21)\\
&=\mathbb{E}\left[\exp\left(\int\left(\mathrm{e}^{-\phi(x,y)}-1\right)Z(dx)\sqrt{2}C\mathrm{e}^{-\sqrt{2}y}dy\right)\right].
\end{aligned}$$

This is the Laplace functional of the process $\widehat{\mathcal{E}}$, which proves Proposition 6.1. □

To prove Theorem 3.1 we need to combine Proposition 6.1 with the results on the genealogical structure of the extremal particles of BBM obtained in [2] and the convergence of the decoration point process Δ (see e.g. Theorem 2.3 of [1]).

Proof of Theorem 3.1. For $x_{i_k}(t)\in\operatorname{supp}\left(\mathcal{E}^{(r_d/t)}(t)\right)$ define the process of recent relatives by

$$\Delta_{t,r}^{(i_k)}=\delta_0+\sum_{j:\tau_j^{i_k}>t-r}\mathcal{N}_j^{i_k}, \tag{6.22}$$

where $\tau_j^{i_k}$ are the branching times along the path $s\mapsto x_{i_k}(s)$ enumerated backwards in time and $\mathcal{N}_j^{i_k}$ the point measures of particles whose ancestor was born at $\tau_j^{i_k}$. In the same way let $\Delta_r^{(i_k)}$ be independent copies of Δ_r which is defined as

$$\Delta_r \equiv \lim_{t\uparrow\infty} \sum_{i=1}^{n(t)} \mathbb{1}_{d\left(\tilde{x}_i(t), \arg\max_{j\leq n(t)} \tilde{x}_j(t)\right)\geq t-r} \delta_{\tilde{x}_i(t)-\max_{j\leq n(t)} \tilde{x}_j(t)}, \tag{6.23}$$

the point measure obtained from Δ by only keeping particles that branched of the maximum after time $t-r$ (see the backward description of Δ in [1]). By Theorem 2.3 of [1] we have that (the labelling i_k refers to the thinned process $\mathcal{E}^{(r_d/t)}(t)$)

$$\left(x_{i_k}(r_d) - \sqrt{2}r_d + M_{i_k}(t-r_d), \Delta_{t,r_d}^{(i_k)}\right)_{1\leq k\leq n^*(t)} \Rightarrow \left(x_j(r_d) - \sqrt{2}r_d + M_j, \Delta_{r_d}^{(j)}\right)_{j\leq n(r_d)}, \tag{6.24}$$

as $t\uparrow\infty$, where M_j are independent copies of M with law ω (see (3.1)). Moreover, $\Delta_{r_d}^{(j)}$ is independent of $(M^{(j)})_{j\leq n(r_d)}$. Looking now at the the Laplace functional for the complete point process $\widetilde{\mathcal{E}}_t$,

$$\widetilde{\Psi}_t(\phi) \equiv \mathbb{E}\left[\mathrm{e}^{\int \phi(x,y)\widetilde{\mathcal{E}}_t(dx,dy)}\right], \tag{6.25}$$

for ϕ as in (6.9), and doing the same manipulations as in the proof of Proposition 6.1, shows that

$$\widetilde{\Psi}_t(\phi) = \mathbb{E}\left[\exp\left(-\sum_{k=1}^{n(t)} \phi\left(\gamma(x_k(r)), \bar{x}_k(t)\right)\right)\right] + O(\epsilon). \tag{6.26}$$

Denote by $\mathcal{C}_{t,r}(D)$ the event

$$\mathcal{C}_{t,r}(D) = \forall i,j \leq n(t) \text{ with } x_i(t), x_j(t) \in D + m(t)\text{: } d(x_i(t), x_j(t)) \notin (r, t-r). \tag{6.27}$$

By Theorem 2.1 in [2] we know that, for each $D\subset\mathbb{R}$ compact,

$$\lim_{r\uparrow\infty} \sup_{t>3r} \mathbb{P}\left((\mathcal{C}_{t,r}(D))^c\right) = 0. \tag{6.28}$$

Hence by introducing $1 = \mathbb{1}_{\left(\mathcal{C}_{t,r}(\mathrm{supp}_y\,\phi)\right)^c} + \mathbb{1}_{\mathcal{C}_{t,r}(\mathrm{supp}_y\,\phi)}$ into (6.26), we obtain that

$$\widetilde{\Psi}_t(\phi) = \mathbb{E}\left[\mathrm{e}^{-\sum_{k=1}^{n^*(t)}\left(\phi\left(\gamma(x_{i_k}(r)), \bar{x}_{i_k}(t)\right) + \sum_j \phi\left(\gamma(x_{i_k}(r)), \bar{x}_{i_k}(t) + \Delta_{t,r_d}^{(i_k,j)}\right)\right)}\right] + O(\epsilon), \tag{6.29}$$

where $\Delta_{t,r_d}^{(i_k,j)}$ are the atoms of $\Delta_{t,r_d}^{(i_k)}$. Hence it suffices to show that

$$\sum_{k=1}^{n^*(t)} \sum_j \delta_{(\gamma(x_{i_k}(r)), \bar{x}_{i_k}(t)) + (0, \Delta_{t,r_d}^{(i_k,j)})} \tag{6.30}$$

converges weakly when first taking the limit $t\uparrow\infty$ and then the limit $r_d\uparrow\infty$ and finally $r\uparrow\infty$. But by (6.24),

$$\lim_{t\uparrow\infty} \sum_{k=1}^{n^*(t)} \sum_\ell \delta_{(\gamma(x_{i_k}(r)), \bar{x}_{i_k}(t)) + (0, \Delta_{t,r_d}^{(i_k,\ell)})} = \sum_{j=1}^{n(r_d)} \sum_\ell \delta_{(\gamma(x_j(r)), x_j(r_d) - \sqrt{2}r_d + M_j) + (0, \Delta_{r_d}^{(j,\ell)})}. \tag{6.31}$$

The limit as first r_d and then r tend to infinity of the process on the right-hand side exists and is equal to $\widetilde{\mathcal{E}}$ by 6.1 (in particular (6.3)). This concludes the proof of Theorem 3.1. $\square$

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CHAPTER 5

The Glassy Phase of the Complex Branching Brownian Motion Energy Model

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ELECTRONIC COMMUNICATIONS in PROBABILITY

The glassy phase of the complex branching Brownian motion energy model*

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Abstract

We identify the fluctuations of the partition function for a class of random energy models, where the energies are given by the positions of the particles of the complex-valued branching Brownian motion (BBM). Specifically, we provide the weak limit theorems for the partition function in the so-called "glassy phase" – the regime of parameters, where the behaviour of the partition function is governed by the extrema of BBM. We allow for arbitrary correlations between the real and imaginary parts of the energies. This extends the recent result of Madaule, Rhodes and Vargas [19], where the uncorrelated case was treated. In particular, our result covers the case of the real-valued BBM energy model at complex temperatures.

Keywords: Gaussian processes; branching Brownian motion; logarithmic correlations; random energy model; phase diagram; extremal processes; cluster processes; multiplicative chaos.
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1 Introduction

Phase transitions arise via an analyticity breaking of the logarithm of the partition function (see, e.g., Ruelle [22]). To analyse this phenomenon, the study of partition functions at *complex temperatures* is of a key interest, as was observed by Lee and Yang [24, 17]. Another motivation to study complex-valued Hamiltonians comes from quantum physics. There, partition functions with complex energies emerge naturally, e.g., from the Schrödinger equation via "imaginary time" Feynman's path integrals.

It is believed that large classes of models of disordered systems fall in the same universality class and, in particular, share the same shape of the phase diagram. Random energy models were proven to be useful in exploring universality classes in mean-field disordered systems, see, e.g., Bovier [6], Panchenko [21] and Kistler [13]. A number of random energy models with complex energies has been considered in the literature. One of the simplest such models (in terms of the correlation structure of the energies) is the so called *Random Energy Model* (REM). For this model, the analyticity of the log-partition function was studied in the seminal work by Derrida [9] and later by

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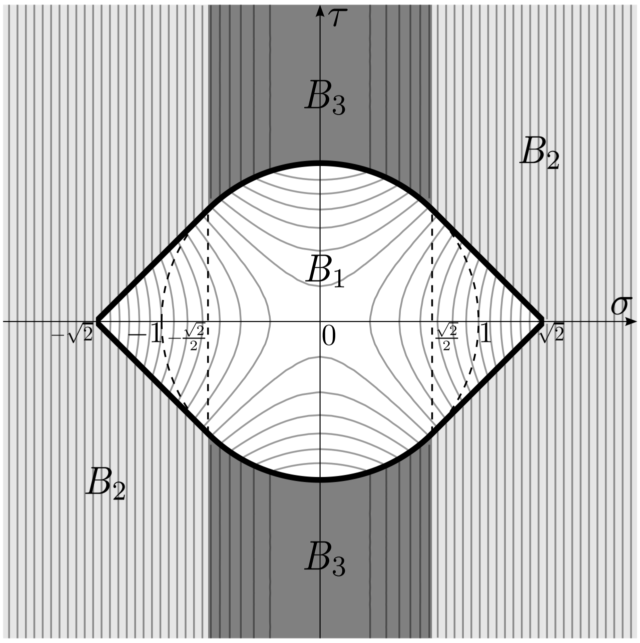

Figure 1: Phase diagram of the REM (and conjecturally of the BBM energy model). The grey curves are the level lines of the limiting log-partition function, cf. (1.18). This paper mainly deals with phase B_2.

Koukiou [15]. The full phase diagram of this model at complex temperatures including the fluctuations and zeros of the partition function were identified by Kabluchko and one of us in [11]. In particular, the case of arbitrary correlations between the imaginary and real parts of the energies was considered in [11]. The same authors answered in [12] similar questions about the *Generalized Random Energy model* (GREM) – a model with hierarchical correlations – and obtained the full phase diagram. In the complex GREM, the phase diagram turned out to have a much richer structure than that of the complex REM. This sheds some light on the phase diagrams of the models beyond the complex REM universality class.

It is known that models with *logarithmic correlations* between the energies are at the borderline of the REM universality class. In particular, they are expected to have the same phase diagram. This has been shown for directed polymers on a tree with complex-valued energies by Derrida, Evans, and Speer [10], and for a model of complex multiplicative cascades by Barral, Jin, and Mandelbrot [5]. Lacoin, Rhodes, and Vargas [16] analysed the phase diagram for complex *Gaussian multiplicative chaos* – a model with logarithmic correlations between the energies on a Euclidean space. There, only the case without correlations between the imaginary and real parts of the energy was treated. It turned out that the phase diagram coincides with the REM one, see Figure 1.

In [16], the analysis of the so-called “glassy” phase B_2, see Figure 1, was left open. In this phase, the partition function is dominated by the extreme values of the energies. Phase B_2 was analysed by Madaule, Rhodes, and Vargas [19] in a continuous model with logarithmic correlations on a tree – the complex *BBM energy model*, but again only when the imaginary and real parts of the energies are uncorrelated. In this model, a deeper understanding of phase B_2 is possible due to recent progress in the analysis of the extremal process of BBM by Aïdékon, Berestycki, Brunet, and Shi [1] and Arguin, Bovier, and Kistler [3]. Madaule, Rhodes, and Vargas [20], have recently analysed the behaviour of the partition function on the boundary between phases B_1 and B_2 (see Figure 1).

In this article, we extend the result of [19]. Specifically, we prove the weak convergence of the (rescaled) partition function of the complex BBM energy model in phase B_2 to a non-trivial distribution. We allow for arbitrary correlations between the real and imaginary parts of the energy. In particular, this covers the complex temperature case, in which the real and imaginary parts of the random energies have maximal correlation (i.e., they are a.s. equal). This case is especially relevant for the Lee-Yang program.

1.1 Branching Brownian motion.

Before stating our results, let us briefly recall the construction of a BBM. Consider a canonical continuous branching process: a *continuous time Galton-Watson* (GW) process [4]. It starts with a single particle at time zero. After an exponential time of parameter one, this particle splits into $k \in \mathbb{Z}_+$ particles according to some probability distribution $(p_k)_{k\geq 0}$ on $\mathbb{Z}_+$. Then, each of the new-born particles splits independently at independent exponential (parameter 1) times again according to the same $(p_k)_{k\geq 0}$, and so on. We assume that $\sum_{k=1}^{\infty} p_k = 1$.[1] In addition, we assume that $\sum_{k=1}^{\infty} kp_k = 2$ (i.e., the expected number of children per particle equals two)[2]. Finally, we assume that $K := \sum_{k=1}^{\infty} k(k-1)p_k < \infty$ (finite second moment)[3]. At time $t = 0$, the GW process starts with just one particle.

For given $t \geq 0$, we label the particles of the process as $i_1(t), \dots, i_{n(t)}(t)$, where $n(t)$ is the total number of particles at time t. Note that under the above assumptions, we have $\mathbb{E}[n(t)] = \mathrm{e}^t$. For $s \leq t$, we denote by $i_k(s,t)$ the unique ancestor of particle $i_k(t)$ at time s. In general, there will be several indices k, l such that $i_k(s,t) = i_l(s,t)$. For $s, r \leq t$, define the time of the most recent common ancestor of particles $i_k(r,t)$ and $i_l(s,t)$ as

$$d(i_k(r,t), i_l(s,t)) := \sup\{u \leq s \wedge r \colon i_k(u,t) = i_l(u,t)\}. \tag{1.1}$$

For $t \geq 0$, the collection of all ancestors naturally induces the random tree

$$\mathbb{T}_t := \{i_k(s,t) \colon 0 \leq s \leq t, 1 \leq k \leq n(t)\} \tag{1.2}$$

called the *GW tree up to time* t. We denote by $\mathcal{F}^{\mathbb{T}_t}$ the σ-algebra generated by the GW process up to time t.

In addition to the genealogical structure, the particles get a *position* in $\mathbb{R}$. Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time when the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the $n(t)$ particles at time $t \geq 0$ by $x_1(t), \dots, x_{n(t)}(t)$ and by $x_1(s,t), \dots, x_{n(t)}(s,t)$ the positions of their ancestors at time $s \geq 0$.

We define BBM as a family of Gaussian processes,

$$x_t := \{x_1(s,t), \dots, x_{n(t)}(s,t) \colon s \leq t\} \tag{1.3}$$

indexed by time horizon $t \geq 0$. Note that conditionally on the underlying GW tree these Gaussian processes have the following covariance

$$\mathbb{E}\left[x_k(s,t)x_l(r,t) \mid \mathcal{F}^{\mathbb{T}_t}\right] = d(i_k(s,t), i_l(r,t)), \quad s, r \in [0,t], \quad k, l \leq n(t). \tag{1.4}$$

[1] This implies that $p_0 = 0$, so none of the particles ever dies.

[2] The latter assumption is just a matter of normalization. Any expected number of children greater than 1 (= the supercritical regime) is allowed and the results of this paper remain valid with appropriate modifications of constants.

[3] Under the stated conditions, the convergence of the extremal process of BBM, on which we rely, is proven in [3]. For the case of branching random walk, using truncation techniques, Madaule [18] has shown the same under conditions that would in the Gaussian case imply finiteness of $\sum_k p_k k (\ln k)^3$. This could probably be carried over to BBM. It is not clear whether the result holds under the Kesten-Stigum condition $\sum_k p_k k \ln k < \infty$. For a discussion on these issues, we refer to the lecture notes by Shi [23]. In the present paper, we are not concerned with improving the conditions on the offspring distribution.

Bramson [7, 8] showed that

$$m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t \tag{1.5}$$

is the order of the maximal position among all BBM particles alive at large time t, i.e.,

$$\lim_{t\uparrow\infty} \mathbb{P}\left\{ \max_{k\le n(t)} x_k(t) - m(t) \le y \right\} = \mathbb{E}\left[\mathrm{e}^{-CZ\mathrm{e}^{-\sqrt{2}y}} \right], \quad y \in \mathbb{R}, \tag{1.6}$$

where $C > 0$ is a constant and Z is the a.s. limit of the so-called *derivative martingale*:

$$Z := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} (\sqrt{2}t - x_k(t))\mathrm{e}^{-\sqrt{2}(\sqrt{2}t - x_k(t))}, \quad \text{a.s.} \tag{1.7}$$

In [1, 3], as $t \uparrow \infty$, the non-trivial limiting point process of the (shifted by $m(t)$) particles of BBM was identified. Specifically, it was shown that the point process,

$$\mathcal{E}_t := \sum_{k=1}^{n(t)} \delta_{x_k(t)-m(t)}, \quad t \in \mathbb{R}_+ \tag{1.8}$$

converges in law as $t \uparrow \infty$ to the point process

$$\mathcal{E} := \sum_{k,l} \delta_{\eta_k + \Delta_l^{(k)}}, \tag{1.9}$$

where:

(a) $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of a Cox process with *random intensity measure* $CZ\mathrm{e}^{-\sqrt{2}y}\mathrm{d}y$, where C and Z are the same as in (1.6).

(b) $\{\Delta_l^{(k)}\}_{l\in\mathbb{N}} \subset \mathbb{R}$ are the atoms of independent and identically distributed point processes $\Delta^{(k)}$, $k \in \mathbb{N}$ called *clusters* which are independent copies of the limiting point process

$$\Delta := \lim_{t\uparrow\infty} \sum_{k=1}^{n(t)} \delta_{\hat{x}_k(t) - \max_{l\le n(t)} \hat{x}_l(t)} \tag{1.10}$$

with $\hat{x}(t)$ being BBM $x(t)$ conditioned on $\max_{k\le n(t)} x_k(t) \ge \sqrt{2}t$.

1.2 Branching Brownian motion energy model at complex temperatures with arbitrary correlations

Let $\rho \in [-1, 1]$. For any $t \in \mathbb{R}_+$, let $X(t) := (x_k(t))_{k\le n(t)}$ and $Y(t) := (y_k(t))_{k\le n(t)}$ be two BBMs with the same underlying GW tree such that, for $k \le n(t)$,

$$\mathrm{Cov}(x_k(t), y_k(t)) = |\rho| t. \tag{1.11}$$

Then,

$$Y(t) \overset{\mathrm{D}}{=} \rho X(t) + \sqrt{1-\rho^2} Z(t), \tag{1.12}$$

where "$\overset{\mathrm{D}}{=}$" denotes equality in distribution and $Z(t) := (z_i(t))_{i\le n(t)}$ is a branching Brownian motion with the same underlying GW process which is independent from $X(t)$. Representation (1.12) allows us to handle arbitrary correlations by decomposing the process Y into a part independent from X and a fully correlated one.

We define the partition function for the complex BBM energy model with correlation ρ at inverse temperature $\beta := \sigma + i\tau \in \mathbb{C}$ by

$$\widetilde{\mathcal{X}}_{\beta,\rho}(t) := \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t) + i\tau y_k(t)}. \tag{1.13}$$

1.3 Main results

Let us specify the three phases depicted on Figure 1 analytically:

$$B_1 := \mathbb{C} \setminus \overline{B_2 \cup B_3}, \quad B_2 := \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 > 1, |\sigma| + |\tau| > \sqrt{2}\}, \\ B_3 := \{\sigma + i\tau \in \mathbb{C} \colon 2\sigma^2 < 1, \sigma^2 + \tau^2 > 1\}. \tag{1.14}$$

In this paper, we focus on the *glassy phase* B_2. We start with the convergence of the partition function in the case of the real BBM energy model at complex temperatures. We say that a complex-valued r.v. Y is *isotropic α-stable* if there exists $c \in \mathbb{R}_+$ and $\alpha \in (0, 2]$ such that

$$\mathbb{E}[\mathrm{e}^{i\mathrm{Re}(\bar{z}Y)}] = \mathrm{e}^{-c|z|^\alpha}, \quad \text{for all } z \in \mathbb{C}. \tag{1.15}$$

Recall the notation from (1.9).

Theorem 1.1 (Partition function fluctuations for $|\rho| = 1$)**.** *For $\beta = \sigma + i\tau \in B_2$, the rescaled partition function $\mathcal{X}_{\beta,1}(t) := \mathrm{e}^{-\beta m(t)} \widetilde{\mathcal{X}}_{\beta,1}(t)$ converges in law to the r.v.*

$$\mathcal{X}_{\beta,1} := \sum_{k,l \geq 1} \mathrm{e}^{\beta\left(\eta_k + \Delta_l^{(k)}\right)}, \quad \textit{as } t \uparrow \infty. \tag{1.16}$$

Theorem 1.2 (Partition function fluctuations for $|\rho| \in (0, 1)$)**.** *For $\beta = \sigma + i\tau \in B_2$ and $|\rho| \in (0, 1)$, the rescaled partition function $\mathcal{X}_{\beta,\rho}(t) := \mathrm{e}^{-\sigma m(t)} \widetilde{\mathcal{X}}_{\beta,\rho}(t)$ converges in law to the r.v. $\mathcal{X}_{\beta,\rho}$, as $t \uparrow \infty$. Conditionally on Z, $\mathcal{X}_{\beta,\rho}$ is a complex isotropic $\sqrt{2}/\sigma$-stable r.v.*

Remark 1.3. For $\rho = 0$, Theorem 1.2 was proven in [19]. Our proof uses a representation of correlated real and imaginary parts in terms of independent BBM's. As in [19], we control second moments. However, the way we do this is different and simpler then the method used in that paper, which relies on decomposing the paths of the BBM particles according to the time and location of the minimal position along the given path. Our approach uses instead the upper envelope for ancenstral paths that was obtained in [2].

Remark 1.4. Note that the fluctuations of the partition function in the complex BBM energy model (cf., Theorems 1.1, 1.2) are governed by the extremal process $\mathcal{E}$. Thus, the fluctuations are different from the ones in the complex REM [11, Theorems 2.8, 2.20] which are governed by a Poisson point process. Despite the differences in fluctuations, we conjecture that in the limit as $t \uparrow \infty$ the *log-partition function*

$$p_t(\beta) := \frac{1}{t} \log |\widetilde{\mathcal{X}}_{\beta,\rho}(t)|, \quad t \in \mathbb{R}_+, \quad \beta \in \mathbb{C} \tag{1.17}$$

of the complex BBM energy model is the same as in the complex REM.

Conjecture 1.5 (Phase diagram)**.** *For any $\rho \in [-1, 1]$, the complex BBM energy model has the same free energy and the phase diagram (cf.,* Figure 1*) as the complex REM, i.e.,*

$$\lim_{t \uparrow \infty} p_t(\beta) =: p(\beta) = \begin{cases} 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in \overline{B_1}, \\ \sqrt{2}|\sigma|, & \beta \in \overline{B_2}, \\ \frac{1}{2} + \sigma^2, & \beta \in \overline{B_3}, \end{cases} \tag{1.18}$$

and the convergence in (1.18) *holds in probability and in L^1.*

Remark 1.6. Convergence in probability for $\beta \in B_2$ in (1.18) follows from Theorems 1.1 and 1.2 by [11, Lemma 3.9 (1)]. The remaining Parts B_1 and B_3 of Conjecture 1.5 are supported by results for similar models, e.g., [10, 5, 16, 11, 12] and by the following intuition.

For $\beta \in B_1$, $\widetilde{\mathcal{X}}_{\beta,\rho}(t)/\mathbb{E}[\widetilde{\mathcal{X}}_{\beta,\rho}(t)]$ is an L^1-convergent complex-valued martingale (as $t \to \infty$) with expectation 1 and a simple computation shows that

$$|\mathbb{E}[\widetilde{\mathcal{X}}_{\beta,\rho}(t)]| = \exp\left(t + \frac{1}{2}t(\sigma^2 - \tau^2)\right). \tag{1.19}$$

See Appendix A for the L^2-martingale convergence in the domain $|\beta| < 1$.

For $\beta \in B_3$, the variance of the partition function of the REM with e^t independent particles equals

$$\mathrm{e}^t \left(\mathbb{E}[\exp(2\sigma x_1(t))] - \exp\left(\frac{1}{2}t(\sigma^2 - \tau^2)\right)\right) \underset{t\uparrow\infty}{\sim} \exp\left(t + 2\sigma^2 t\right), \tag{1.20}$$

cf. [11]. Therefore, as $t \uparrow \infty$, the standard deviation has a greater order of magnitude than the expectation (1.19). So, in view of the central limit theorem, it is plausible that

$$\widetilde{\mathcal{X}}_{\beta,\rho}(t)/\exp\left(\frac{1}{2}t + \sigma^2 t\right) \tag{1.21}$$

converges as $t \uparrow \infty$ in distribution. However, due to correlations between the particle positions of BBM, the limiting distribution in (1.21) need not be Gaussian, cf. [16, Theorems 4.2 and 6.6] and [11, Eq. (2.11)].

Organization of the rest of the paper. The proofs of Theorems 1.1 and 1.2 consist of two main steps. First, we show that only the extremal particles can contribute to the partition function in the limit as $t \uparrow \infty$ (cf., Proposition 2.1 and its proof in Section 3). Second, we use the continuous mapping theorem to deduce Theorems 1.1 and 1.2 from the behaviour of the extremal process. This is done in Section 2.

2 Convergence of the partition function

First, we state that in the glassy phase B_2 only the extremal particles can contribute to the limit of the partition function as t tends to infinity.

Proposition 2.1. *If $|\rho| \in (0,1]$ and $\beta \in B_2$, then, for all $\delta, \epsilon > 0$, there exists $A_0 > 0$ such that, for all $A > A_0$ and all t sufficiently large,*

$$\mathbb{P}\left\{\left|\sum_{k=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)} \mathbb{1}_{\{x_k(t)-m(t)<-A\}}\right| > \delta\right\} < \epsilon. \tag{2.1}$$

The proof of Proposition 2.1 is postponed until Section 3. Using Proposition 2.1 together with the continuous mapping theorem, we now prove Theorem 1.1.

Proof of Theorem 1.1. Denote by $\mathbb{M}$ the space of locally finite counting measures on $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. We endow $\mathbb{M}$ with the vague topology. Consider for $A \in \mathbb{R}_+$ the functional $\Phi_{\beta,A} \colon \mathbb{M} \to \mathbb{R}$. This functional maps a locally finite counting measure $\zeta = \sum_{i\in I} \delta_{x_i}$ to $\Phi_{\beta,A}(\zeta) := \sum_{i\in I} \mathrm{e}^{\beta x_i} \mathbb{1}_{\{x_i > -A\}}$, where I is a countable index set. The set of locally finite measures ζ on which the functional $\Phi_{\beta,A}$ is not continuous (i.e., ζ charging $-A$ or $+\infty$) has zero measure w.r.t. the law of $\mathcal{E}$. Hence, by the continuous mapping theorem, it follows that $\Phi_{\beta,A}(\mathcal{E}_t)$ converges in law to $\Phi_{\beta,A}(\mathcal{E})$, which is equal to

$$\sum_{k,l\geq 1} \mathrm{e}^{\beta\left(\eta_k + \Delta_l^{(k)}\right)} \mathbb{1}_{\{\eta_k + \Delta_l^{(k)} \geq -A\}}. \tag{2.2}$$

Note that by Proposition 2.1, for all $\epsilon > 0$ and $\delta > 0$, there exists A_0 such that, for all $A > A_0$ and all t sufficiently large,

$$\mathbb{P}\left\{|\mathcal{X}_{\beta,1}(t) - \Phi_{\beta,A}(\mathcal{E}_t)| > \delta\right\} < \epsilon. \tag{2.3}$$

Hence, by Slutsky's Theorem (see, e.g., [14, Theorem 13.18]), $\mathcal{X}_{\beta,1}(t)$ converges in law to

$$\lim_{A\uparrow\infty}\sum_{k,l\geq 1} e^{\beta\left(\eta_k+\Delta_l^{(k)}\right)}\mathbb{1}_{\{\eta_k+\Delta_l^{(k)}\geq -A\}} \tag{2.4}$$

which is equal to $\mathcal{X}_{\beta,1}$. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Using Representation (1.12), we have that $\mathcal{X}_{\beta,\rho}(t)$ is in distribution equal to

$$\sum_{k=1}^{n(t)} \mathrm{e}^{(\sigma+i\rho\tau)(x_k-m(t))+i\sqrt{1-\rho^2}\tau z_k(t)-i\rho\tau m(t)}, \tag{2.5}$$

where $(z_k(t), k\leq n(t))$ are the particles from a BBM that is independent from $X(t)$ (but with respect to the same GW tree). If $|\rho|\neq 1$, then by [19, see Lemma 3.2 and the subsequent discussion before Eq. (3.7) therein] we get that

$$G(t) := \sum_{k=1}^{n(t)} \delta_{(x_k(t)-m(t),\exp(i\sqrt{1-\rho^2}\tau z_k(t)-i\rho\tau m(t)))} \tag{2.6}$$

converges weakly as $t\uparrow\infty$ to

$$\mathcal{G} := \sum_{k,l\geq 1}\delta_{(p_k+\Delta_l^{(k)},U^{(k)}\widetilde{W}_l^{(k)})}, \tag{2.7}$$

where $\left(U^{(k)}\right)_{k\geq 1}$ are i.i.d. uniformly distributed on the unit circle and $\widetilde{W}_l^{(k)}$ are the atoms of a point process on the unit circle. The description of $\widetilde{W}^{(k)}$ could be made more explicit using the description of the cluster process Δ obtained in [1, Theorem 2.3] that encodes the genealogical structure of Δ.

Denote by $\widetilde{\mathbb{M}}$ the space of locally finite counting measures on $\overline{\mathbb{R}}\times\{z\in\mathbb{C}\colon |z|=1\}$. We endow $\widetilde{\mathbb{M}}$ with the (Polish) topology of vague convergence. For $A\in\mathbb{R}_+$, consider the functional $\widetilde{\Phi}_{\beta,A}\colon\widetilde{\mathbb{M}}\to\mathbb{C}$ that maps a locally finite counting measure $\tilde{\zeta}=\sum_{k\in I}\delta_{(x_k,z_k)}$ to $\widetilde{\Phi}_{\beta,A}(\zeta) := \sum_{k\in I}\mathrm{e}^{\beta x_k}z_k\mathbb{1}_{\{x_k>-A\}}$, where I is a countable index set. The set of locally finite measures ζ on which the functional $\Phi_{\beta,A}$ is not continuous (i.e., $\tilde{\zeta}$ charging $(-A,\cdot)$ or $(+\infty,\cdot)$) has zero measure w.r.t. the law of $\mathcal{G}$. Hence, by the continuous mapping theorem, it follows that $\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G}_t)$ converges in law to $\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G})$, which is equal to

$$\sum_{k,l\geq 1}\mathrm{e}^{(\sigma+i\rho\tau)\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_i+\Delta_l^{(k)}\geq -A\}}. \tag{2.8}$$

Since $\mathrm{e}^{(i\rho\tau)\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}$ is also uniformly distributed on the unit circle, (2.8) is equal in distribution to

$$\sum_{k,l\geq 1}\mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_i+\Delta_l^{(k)}\geq -A\}}. \tag{2.9}$$

Note that again by Proposition 2.1, for all $\epsilon>0$ and $\delta>0$, there exists A_0 such that, for all $A>A_0$ and all t sufficiently large,

$$\mathbb{P}\left\{\left|\mathcal{X}_{\beta,\rho}(t)-\widetilde{\Phi}_{\sigma+i\rho\tau,A}(\mathcal{G}_t)\right|>\delta\right\}<\epsilon. \tag{2.10}$$

Hence, by Slutsky's theorem (see, e.g., [14, Theorem 13.18]), $\mathcal{X}_{\beta,\rho}(t)$ converges in law to

$$\lim_{A\uparrow\infty}\sum_{k,l\geq 1} \mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}\mathbb{1}_{\{\eta_k+\Delta_l^{(k)}\geq -A\}} = \sum_{k,l\geq 1}\mathrm{e}^{\sigma\left(\eta_k+\Delta_l^{(k)}\right)}U^{(k)}\widetilde{W}_l^{(k)}. \tag{2.11}$$

We rewrite (2.11) as

$$\sum_{k\geq 1}\mathrm{e}^{\sigma\eta_k}U^{(k)}W^{(k)}, \tag{2.12}$$

where $W^{(k)} := \sum_l \mathrm{e}^{\sigma\Delta_l^{(k)}}\widetilde{W}_l^{(k)}$, $k\geq 1$ are i.i.d. r.v.'s. From (2.12), it follows that conditionally on Z, the distribution of $\mathcal{X}_{\beta,\rho}$ is complex isotropic $\sqrt{2}/\sigma$-stable.

□

3 Proof of Proposition 2.1

Due to symmetry, we only prove Proposition 2.1 for $\sigma,\tau>0$. In the proof of Proposition 2.1, we distinguish two cases:

$$\textbf{(a)}\quad \sigma>\sqrt{2};\qquad \textbf{(b)}\quad \sqrt{2}/2<\sigma\leq\sqrt{2}\text{ and }\sigma+\tau>\sqrt{2}. \tag{3.1}$$

Case (a). In this case, the proof works as in the independent case treated in [19, Lemma 3.5]. For completeness, we also provide the proof in this case. We use a first moment computation together with the upper bound on the maximal position of all particles obtained in [2, Theorem 2.2].

Proof of Proposition 2.1 *in case* (a). Recall the notation from (1.3). By [2, Theorem 2.2], for $0<\gamma<\frac{1}{2}$, there exists $r_\epsilon>0$ such that for all $r>r_\epsilon$ and $t>3r$

$$\mathbb{P}\left\{\exists k\leq n(t)\colon x_k(s,t)>U_{t,\gamma}\text{ for some }s\in[r,t-r]\right\}<\frac{\epsilon}{2}, \tag{3.2}$$

where $U_{t,\gamma}(s) := \frac{s}{t}m(t)+(s\wedge(t-s))^\gamma$. Define the following set on the path space

$$\mathcal{U}_{t,r,\gamma} := \{x(\cdot)\in C(\mathbb{R}_+,\mathbb{R})\colon x(s,t)\leq\frac{s}{t}m(t)+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\}. \tag{3.3}$$

By (3.2), to show (2.1), it suffices to check that, for sufficiently large $A>0$,

$$\mathbb{P}\left\{\left|\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right|>\delta\right\}<\epsilon/2. \tag{3.4}$$

By Markov's inequality, the probability in (3.4) is bounded from above by

$$\begin{aligned}&\frac{1}{\delta}\mathbb{E}\left[\left|\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right|\right]\\ &\leq\frac{1}{\delta}\mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right].\end{aligned} \tag{3.5}$$

We rewrite the expectation in the r.h.s. of (3.5) as $\sum_{B>A}S(B,t)$, where

$$S(B,t) := \mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\right]. \tag{3.6}$$

Next, we manipulate the event

$$\begin{aligned}&\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}\\&\subset\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{\xi(s)\le\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\},\end{aligned}\tag{3.7}$$

where $\xi_k(s) := x_k(s,t) - \frac{s}{t}x_k(t)$ is a Brownian bridge from 0 to 0 in time t that is independent from $x_k(t)$. Hence, we can bound $S(B,t)$ from above by

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))}\mathbb{1}_{\{x_k(t)-m(t)\in(-B+1,-B]\}\cap\{\xi_k(s)\le\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\}}\right]\\&=\mathrm{e}^t\mathbb{E}\left[\mathrm{e}^{\sigma(x(t)-m(t))}\mathbb{1}_{x(t)-m(t)\in(-B+1,-B]}\right]\mathbb{P}\left\{\xi(s)\le\frac{s}{t}B+(s\wedge(t-s))^\gamma,\forall s\in[r,t-r]\right\},\end{aligned}\tag{3.8}$$

where $x(t)$ is normal distributed with mean 0 and variance t and $\xi(\cdot)$ is a Brownian bridge from 0 to 0 in time t independent from $x(t)$. The expectation in the second line of (3.8) is equal to

$$\int_{m(t)-B}^{m(t)-B+1}\mathrm{e}^{\sigma(x-m(t))}\mathrm{e}^{-x^2/2t}\frac{\mathrm{d}x}{\sqrt{2\pi t}}=\mathrm{e}^{-\sigma m(t)+\frac{\sigma^2 t}{2}}\int_{m(t)-B-\sigma t}^{m(t)-B+1-\sigma t}\mathrm{e}^{-w^2/2t}\frac{\mathrm{d}w}{\sqrt{2\pi t}},\tag{3.9}$$

where we changed variables $x = w + \sigma t$. Since $\sigma > \sqrt{2}$, by the definition of $m(t)$ it holds that $m(t) - B - \sigma t < (\sqrt{2} - \sigma)t < 0$, for all $t > 1$. Therefore, using the standard Gaussian tail bound,

$$\int_{-\infty}^{-x}\mathrm{e}^{-w^2/2}\frac{\mathrm{d}w}{\sqrt{2\pi}}\le\frac{1}{\sqrt{2\pi}x}\mathrm{e}^{-x^2/2},\quad x>0,\tag{3.10}$$

we can bound (3.9) using $m^2(t) = 2t - 3t\log t + \left(3\log t/(2\sqrt{2})\right)^2$ from above by

$$\frac{\sqrt{t}}{\sqrt{2\pi}(B-1+\sigma t-m(t))}\mathrm{e}^{-\sigma m(t)+\frac{\sigma^2 t}{2}}\mathrm{e}^{-(m(t)-B+1-\sigma t)^2/2t}\underset{t\uparrow\infty}{\sim}\frac{t}{\sqrt{2\pi}(\sigma-\sqrt{2})}\mathrm{e}^{-t+(\sqrt{2}-\sigma)(B-1)}.\tag{3.11}$$

Next, we analyse the probability in the r.h.s. of (3.8). We bound it, for $B < t^\gamma/3$, from above by

$$\mathbb{P}\left\{\xi(s)\le 2(s\wedge(t-s))^\gamma,\forall s\in[r\vee B^{1/\gamma},(t-B^{1/\gamma})\wedge(t-r)]\right\}.\tag{3.12}$$

By the proof of [2, Theorem 2.3, see (5.55)], for all r large enough, probability (3.12) is bounded from above by

$$\mathbb{P}\left\{\xi(s)\le 0,\forall s\in[r\vee B^{1/\gamma},(t-B^{1/\gamma})\wedge(t-r)]\right\}(1+\epsilon)\le\frac{2(B^{1/\gamma}\wedge r)}{t-2(B^{1/\gamma}\wedge r)}(1+\epsilon),\tag{3.13}$$

where in the last step we used [2, Lemma 3.4]. Plugging the estimates from (3.11) and (3.13) into (3.8), we get

$$S(B,t)\le\left(\frac{2(B^{1/\gamma}\vee r)}{t-2(B^{1/\gamma}\vee r)}(1+\epsilon)\mathbb{1}_{\{B>t^\gamma/3\}}+\mathbb{1}_{\{B\le t^\gamma/3\}}\right)\frac{t\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma-\sqrt{2})}(1+o(1)).\tag{3.14}$$

Note that in (3.14) and below $o(1)$ denotes a t-dependent non-random quantity with

$$o(1)\underset{t\uparrow\infty}{\longrightarrow}0.\tag{3.15}$$

From (3.14) follows that $\lim_{t\uparrow\infty}\sum_{B>t/3}S(B,t)=0$ and

$$\sum_{B=A+1}^{t^\gamma/3} S(B,t) \leq \sum_{B=A+1}^{t^\gamma/3} \frac{2t(B^{1/\gamma}\vee r)\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma-\sqrt{2})(t-2(B^{1/\gamma}\vee r))}(1+\epsilon), \tag{3.16}$$

which can be made smaller than $\epsilon/2$ by taking A large enough since $\sqrt{B^{1/\gamma}\wedge r}\mathrm{e}^{(\sqrt{2}-\sigma)(B-1)}$ is summable in B (because $\sqrt{2}-\sigma<0$). This concludes the proof of Theorem 2.1 in case (a). □

Case (b). In this case, the analysis is somewhat more intricate and we have to employ the imaginary part of the energy.

Short outline of the proof. To prove (2.1), we first apply the Chebyshev inequality to the absolute value of the truncated partition function. Then, we compute the second moment which arises in the Chebyshev inequality. Along the way, we first use Representation (1.12) and compute the expectation w.r.t. $z(t)$ conditionally on $\mathcal{F}^{\mathbb{T}_t}$, see (3.19). Starting from (3.22), we use the so-called upper envelope for the given path of $x(t)$ (see [2, Theorem 2.2]) to control the expectation w.r.t. $x(t)$. Technically, we have to distinguish between three regimes for the time of the most recent common ancestor $q_{k,l}=d(x_k(t),x_l(t))$. The corresponding terms are controlled separately starting from Eq. (3.35).[4]

Proof of Proposition 2.1 in case (b). We proceed as in case (a) until (3.4). This time, using Chebyshev's inequality, we bound the probability in (3.4) by

$$\frac{1}{\delta^2}\mathbb{E}\left[\Big|\sum_{k=1}^{n(t)}\mathrm{e}^{\sigma(x_k(t)-m(t))+i\tau y_k(t)}\mathbb{1}_{\{x_k(t)-m(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\Big|^2\right], \tag{3.17}$$

We introduce the shorthand notation $\widetilde{x}_k(t):=x_k(t)-m(t)$, $k\leq n(t)$. Using this notation, together with Representation (1.12), we get that (3.17) is equal to

$$\frac{1}{\delta^2}\mathbb{E}\left[\Big|\sum_{k=1}^{n(t)}\mathrm{e}^{(\sigma+i\rho\tau)x_k(t)-\sigma m(t)+i\sqrt{1-\rho^2}\tau z_k(t)}\mathbb{1}_{\{\widetilde{x}_k(t)<-A\}\cap\{x_k\in\mathcal{U}_{t,r,\gamma}\}}\Big|^2\right]. \tag{3.18}$$

Define $\lambda:=\sigma+i\rho\tau$. Observe that $|z|^2=z\bar{z}$, for $z\in\mathbb{C}$. Hence, the expectation in (3.18) is equal to

$$\mathbb{E}\left[\sum_{k,l=1}^{n(t)}\mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)+i\sqrt{1-\rho^2}\tau(z_l(t)-z_k(t))}\mathbb{1}_{\forall_{j\in\{l,k\}}(\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\})}\right] \tag{3.19}$$

$$= \mathbb{E}\left[\sum_{k,l=1}^{n(t)}\Big(\mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)}\mathbb{1}_{\forall_{j\in\{l,k\}}(\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\})} \times\mathbb{E}\left[\mathrm{e}^{i\sqrt{1-\rho^2}\tau(z_l(t)-z_k(t))}\mid\mathcal{F}^{\mathbb{T}_t}\right]\Big)\right], \tag{3.20}$$

where we used that $(z_k(t),k\leq n(t))$ is, conditionally on $\mathbb{T}_t$, independent from $(x_k(t),k\leq n(t))$. Since $(z_k(t),k\leq n(t))$ is a BBM on the same GW tree as x, (3.19) is equal to

$$\mathbb{E}\left[\sum_{k,l=1}^{n(t)}\mathrm{e}^{\overline{\lambda}x_l(t)+\lambda x_k(t)-2\sigma m(t)+(1-\rho^2)\tau^2(t-d(x_l(t),x_k(t)))}\mathbb{1}_{\forall_{j\in\{l,k\}}\{\widetilde{x}_j(t)<-A\}\cap\{x_j\in\mathcal{U}_{t,r,\gamma}\}}\right]. \tag{3.21}$$

[4] Note that this approach to control the second moment differs from the one used in [19]. The latter one relies on decomposing the paths of the BBM particles according to the time and location of the minimal position along the given path.

We introduce the time of the most recent common ancestor $q_{k,l} = d(x_k(t), x_l(t))$, where $d(\cdot,\cdot)$ is defined in (1.1), and rewrite (3.21) as $\sum_{B>1} T(B,t)$, where

$$T(B,t) := \mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda} x_l(t)+\lambda x_k(t)-2\sigma m(t)} \mathrm{e}^{(1-\rho^2)\tau^2(t-q_{k,l})} \mathbb{1}_{\mathcal{U}^{l,k}_{B,q,t}}\right], \tag{3.22}$$

and

$$\begin{aligned} \mathcal{U}^{l,k}_{B,q,t} &:= \cap_{j\in\{l,k\}}\{\widetilde{x}_j(t) < -A\} \cap \{x_j(s) \le U_{t,\gamma}(s), \forall s \in [r, t-r]\} \\ &\cap \{x_j(q_{k,l}) - U_{t,\gamma}(q_{k,l}) \in [-B+1, -B]\}. \end{aligned} \tag{3.23}$$

Similar to (3.7), we now relax conditions on the path of the particle. If $q_{k,l} > \frac{3}{4}t$, then we get

$$\begin{aligned} \mathcal{U}^{l,k}_{B,q,t} \subset \cap_{j\in\{l,k\}} &\{\widetilde{x}_j(t) < -A\} \cap \{x_l(q_{k,l}, t) - U_{t,\gamma}(q_{k,l}) \in [-B+1, -B]\} \\ &\cap \{\xi^q_l(s) \le 8(s \wedge (q_{k,l}-s))^\gamma, \forall s \in [B^{1/\gamma} \vee r, q_{k,l} - (B^{1/\gamma} \wedge r)]\} =: \mathcal{T}^{l,k}_{B,q,t}, \end{aligned} \tag{3.24}$$

where $\xi^q_l(s) := x_l(s,t) - \frac{s}{q} x_l(q_{k,l}, t)$ is a Brownian bridge from 0 to 0 in time $q_{k,l}$, which is, in particular, independent of $x_l(q_{k,l}, t)$. Moreover, for $q \le \frac{3}{4}t$, we have

$$\mathcal{U}^{l,k}_{B,q,t} \subset \cap_{j\in\{l,k\}}\{\widetilde{x}_j(t) < -A\} \cap \{x_l(q_{k,l}, t) - U_{t,\gamma}(q_{k,l}) \in [-B+1, -B]\} =: \mathcal{S}^{l,k}_{B,q,t}. \tag{3.25}$$

Hence, $T(B,t)$ defined in (3.22) is bounded from above by

$$\begin{aligned} &\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\overline{\lambda} x_l(t)+\lambda x_k(t)-2\sigma m(t)} \mathrm{e}^{(1-\rho^2)\tau^2(t-q_{k,l})} \left(\mathbb{1}_{\{q_{k,l}>\frac{3}{4}t\}\cap\mathcal{T}^{l,k}_{B,q,t}} + \mathbb{1}_{\{q_{k,l}\le\frac{3}{4}t\}\cap\mathcal{S}^{l,k}_{B,q,t}}\right)\right] \\ &= K\int_0^t \mathrm{d}q\, \mathrm{e}^{2t-q+(1-\rho^2)\tau^2(t-q)} \int_{U_{t,\gamma}(q)-B}^{U_{t,\gamma}(q)-B+1} \mathrm{d}x \int_{-\infty}^{m(t)-A-x} \mathrm{d}y \int_{-\infty}^{m(t)-A-x} \mathrm{d}y' \\ &\quad\times \mathrm{e}^{\sigma(2x+y+y'-2m(t))+i\rho\tau(y'-y)} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \frac{1}{2\pi(t-q)} \mathrm{e}^{-\frac{x^2}{2q}} \frac{1}{\sqrt{2\pi q}} \\ &\quad\times \left(\mathbb{1}_{\{q\le\frac{3}{4}t\}} + \mathbb{1}_{\{q\ge\frac{3}{4}t\}} \mathbb{P}\left\{\xi^q(s) \le 8(s\wedge(q-s))^\gamma, \forall s \in [B^{1/\gamma}\vee r, q - B^{1/\gamma}\wedge r]\right\}\right), \end{aligned} \tag{3.26}$$

where $K = \sum_{k=1}^\infty k(k-1)p_k$. It is in (3.26) that we need the second moment assumption on the distribution $(p_k)_{k\ge0}$, cf. Footnote 3. First, observe that, for $B < t^\gamma/3$, as in (3.13), the probability in (3.26) is bounded from above by $\frac{2(B^{1/\gamma}\vee r)}{q-2(B^{1/\gamma}\vee r)}(1+\epsilon)$. Observe that $m(t) - A - x \le m(t) - A - U_{t,\gamma}(q) + B$. We compute first the integrals with respect to y and y' in (3.26), i.e.,

$$\int_{-\infty}^{\mathcal{D}_{A,B,q}} \int_{-\infty}^{\mathcal{D}_{A,B,q}} \mathrm{e}^{\sigma(2x+y+y'-2m(t))+i\rho\tau(y'-y)} \mathrm{e}^{-\frac{y^2+y'^2}{2(t-q)}} \frac{\mathrm{d}y\mathrm{d}y'}{2\pi(t-q)}, \tag{3.27}$$

where $\mathcal{D}_{A,B,q} := m(t) - A - U_{t,\gamma}(q) + B$. We make the following change of variables

$$y = w + \lambda(t-q) \quad \text{and} \quad y' = w' + \overline{\lambda}(t-q). \tag{3.28}$$

Hence, (3.27) is equal to

$$\mathrm{e}^{2\sigma(x-m(t))+(\sigma^2-(\rho\tau)^2)(t-q)} \int_{-\infty}^{\mathcal{D}_{A,B,q}-\lambda(t-q)} \int_{-\infty}^{\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)} \mathrm{e}^{-\frac{w^2+w'^2}{2(t-s)}} \frac{\mathrm{d}w\mathrm{d}w'}{2\pi(t-q)}. \tag{3.29}$$

Using (3.10), we bound (3.29) from above by

$$\mathrm{e}^{2\sigma(x-m(t))+(\sigma-\tau^2)(t-q)}\Big(\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}} + \exp\left(-\frac{(\mathcal{D}_{A,B,q}-\lambda(t-q))^2+\big(\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)\big)^2}{2(t-q)}\right)\mathbb{1}_{\{\mathcal{D}_{A,B,q}\leq\sigma(t-q)\}}\Big). \quad (3.30)$$

Next we carry out the integration over x in (3.26). Note that

$$\int_{U_{t,\gamma}(q)-B}^{U_{t,\gamma}(q)-B+1}\mathrm{e}^{2\sigma x}\mathrm{e}^{-\frac{x^2}{2q}}\frac{\mathrm{d}x}{\sqrt{2\pi q}} = \mathrm{e}^{2\sigma^2 q}\int_{U_{t,\gamma}(q)-B-2\sigma q}^{U_{t,\gamma}(q)-B+1-2\sigma q}\mathrm{e}^{-\frac{v^2}{2q}}\frac{\mathrm{d}v}{\sqrt{2\pi q}}, \quad (3.31)$$

where we made the change of variables $x = v + 2\sigma q$. Observe that $U_{t,\gamma}(q) - 2\sigma q \leq (\sqrt{2}-2\sigma)q < 0$, since $\sigma \geq \frac{1}{\sqrt{2}}$. Therefore, using (3.10), the right-hand side of (3.31) is bounded from above by

$$\frac{\sqrt{q}}{2\sigma q - U_{t,\gamma}(q)+B}\mathrm{e}^{2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}. \quad (3.32)$$

Using the bounds (3.32) and (3.30) in (3.26), we get that (3.26) is bounded from above by

$$\begin{aligned} K\int_0^t &\frac{\sqrt{q}\mathrm{e}^{2t-q+2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}}{2\sigma q - U_{t,\gamma}(q)+B}\mathrm{e}^{-2\sigma m(t)+(\sigma^2-\tau^2)(t-q)} \\ &\times\Big(\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}} + \mathrm{e}^{-\frac{\left(\mathcal{D}_{A,B,q}-\lambda(t-q)\right)^2+\left(\mathcal{D}_{A,B,q}-\overline{\lambda}(t-q)\right)^2}{2(t-q)}}\mathbb{1}_{\{\mathcal{D}_{A,B,q}\geq\sigma(t-q)\}}\Big) \\ &\times\Big(\mathbb{1}_{\{q\leq\frac{3}{4}t\}} + \mathbb{1}_{\{q\geq\frac{3}{4}t,\ B<t^\gamma/3\}}\frac{2(B^{1/\gamma}\vee r)}{q-2(B^{1/\gamma}\vee r)}(1+\epsilon)\Big)\,\mathrm{d}q. \end{aligned} \quad (3.33)$$

Using that $U_{t,\gamma}(q) - 2\sigma q = (\sqrt{2}-2\sigma)q - \frac{q}{t}\frac{3}{2\sqrt{2}}\log t + (q\wedge(t-q))^\gamma$, we start to simplify (3.33). We get

$$\begin{aligned} &\mathrm{e}^{2t-q}\mathrm{e}^{2\sigma^2 q}\mathrm{e}^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q}\mathrm{e}^{-2\sigma m(t)+\frac{(\sigma^2-\tau^2)(t-q)}{2}} \\ &\underset{t\uparrow\infty}{\sim}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)+\left(\frac{3\sigma}{\sqrt{2}}+\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}\right)\log t-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma+(\sqrt{2}-2\sigma)B}. \end{aligned} \quad (3.34)$$

Note that by assumption on σ and τ we have $(\sigma-\sqrt{2})^2-\tau^2 < 0$ and $\sqrt{2}-2\sigma < 0$. Cutting the domain of integration in (3.33) into three parts $q \in [0, t-\log(t)^\alpha]$, $q \in (t-\log(t)^\alpha, t-\frac{A}{2}]$ and $q \in (t-\frac{A}{2}, t]$, for some fixed $\alpha > 1$, we get the following three terms

$$K\int_0^t \ldots\mathrm{d}q = K\left(\int_0^{t-\log(t)^\alpha} + \int_{t-\log(t)^\alpha}^{t-\frac{A}{2}} + \int_{t-\frac{A}{2}}^t\right)\ldots\mathrm{d}q =: K\left((\mathrm{I1})+(\mathrm{I2})+(\mathrm{I3})\right). \quad (3.35)$$

We bound (I1) from above by

$$\begin{aligned} &\int_0^{t-\log(t)^\alpha}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)+\left(\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}+\frac{3\sigma}{\sqrt{2}}\right)\log t-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma+(\sqrt{2}-2\sigma)B}\mathrm{d}q(1+o(1)) \\ &\leq \mathrm{e}^{(\sqrt{2}-2\sigma)B+\frac{3\sigma}{\sqrt{2}}\log t}\int_0^{t-\log(t)^\alpha}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)-(\sqrt{2}-2\sigma)(q\wedge(t-q))^\gamma}\mathrm{d}q(1+o(1)) \\ &\leq \mathrm{e}^{(\sqrt{2}-2\sigma)B}\mathrm{e}^{C\log(t)^\alpha\left((\sigma-\sqrt{2})^2-\tau^2\right)+\frac{3\sigma}{\sqrt{2}}\log t-(\sqrt{2}-2\sigma)\log(t)^{\gamma\alpha}}, \quad t\uparrow\infty, \end{aligned} \quad (3.36)$$

for some constant $C > 0$. Hence,

$$K\sum_{B>1}(\mathrm{I1}) \leq K\mathrm{e}^{C\log(t)^\alpha\left((\sigma-\sqrt{2})^2-\tau^2\right)+\frac{3\sigma}{\sqrt{2}}\log t-(\sqrt{2}-2\sigma)\log(t)^{\gamma\alpha}}\sum_{B>1}\mathrm{e}^{(\sqrt{2}-2\sigma)B}, \quad (3.37)$$

since $\sqrt{2}-2\sigma<0$, we have $\sum_{B>1}\mathrm{e}^{(\sqrt{2}-2\sigma)B}<\infty$. Hence, we can choose t_0 such that, for all $t>t_0$, the r.h.s. of (3.37) less than $\frac{\epsilon}{6}$. For $q\in(t-\log(t)^\alpha,t]$, we observe first that

$$\mathrm{e}^{\left(\frac{(\sqrt{2}-2\sigma)3q}{2\sqrt{2}t}+\frac{3\sigma}{\sqrt{2}}\right)\log t}\underset{t\uparrow\infty}{\sim}\mathrm{e}^{\frac{3}{2}\log t}, \tag{3.38}$$

and, moreover,

$$\frac{2\sqrt{q}(B^{1/\gamma}\vee r)}{(2\sigma q-U_{t,\gamma}(q)+B)\left(q-2(B^{1/\gamma}\vee r)\right)}\le C'\frac{2(B^{1/\gamma}\vee r)}{\sqrt{t}(t-2(B^{1/\gamma}\vee r))}, \tag{3.39}$$

for some constant $C'>0$. Using (3.38) and (3.39), we bound (I2) from above by

$$\begin{aligned}&\int_{t-\log(t)^\alpha}^{t-\frac{A}{2}}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)-(\sqrt{2}-2\sigma)(t-q)^\gamma+(\sqrt{2}-2\sigma)B}C't\mathrm{d}q\\&\qquad\qquad\times\left(\tfrac{2(B^{1/\gamma}\vee r)}{(t-2(B^{1/\gamma}\vee r))}\mathbb{1}_{\{B<t^\gamma/3\}}+\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)(1+o(1))\\&\le C_2\mathrm{e}^{\frac{A}{2}((\sigma-\sqrt{2})^2-\tau^2)}\mathrm{e}^{(\sqrt{2}-2\sigma)B}\left((B^{1/\gamma}\vee r)\mathbb{1}_{\{B<t^\gamma/3\}}+t\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)(1+o(1)),\end{aligned} \tag{3.40}$$

as $t\uparrow\infty$. Using (3.40), we get that $K\sum_{B>1}$(I2) is bounded from above by

$$KC_2\mathrm{e}^{\frac{A}{2}((\sigma-\sqrt{2})^2-\tau^2)}\left(\sum_{B=1}^{[t^\gamma/3]}\mathrm{e}^{(\sqrt{2}-2\sigma)B}(B^{1/\gamma}\vee r)+\sum_{B>[t^\gamma/3]}\mathrm{e}^{(\sqrt{2}-2\sigma)B}t\right)(1+o(1)), \tag{3.41}$$

as $t\uparrow\infty$. Again, since $2-2\sigma<0$, we have $\sum_{B>1}B^{\frac{1}{\gamma}}\mathrm{e}^{(\sqrt{2}-2\sigma)B}<\infty$ and $(\sigma-\sqrt{2})^2-\tau^2<0$. Hence, there exist t_1 and A_1 such that, for all $t>t_1$ and all $A>A_1$, we have that (3.41)$\le\frac{\epsilon}{6}$. Since $\mathcal{D}_{A,B,q}-\sigma(t-q)<0$ for $t-q\le\frac{A}{\sqrt{2}}$ and $B\le\frac{A}{2}$, we bound (I3) from above by

$$\begin{aligned}&\int_{t-\frac{A}{2}}^{t}\mathrm{e}^{(t-q)\left((\sigma-\sqrt{2})^2-\tau^2\right)}\mathrm{e}^{-(\sqrt{2}-2\sigma)(t-q)^\gamma+(\sqrt{2}-2\sigma)B}C't\left(\tfrac{2(B^{1/\gamma}\vee r)}{(t-2(B^{1/\gamma}\vee r))}\mathbb{1}_{\{B<t^\gamma/3\}}+\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)\\&\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{(t-q)}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right)\mathrm{d}q,\quad t\uparrow\infty.\end{aligned} \tag{3.42}$$

Using that $(\sigma-\sqrt{2})^2-\tau^2<0$ and $\sqrt{2}-2\sigma<0$, we bound (3.42) from above by

$$\begin{aligned}&\int_{t-\frac{A}{2}}^{t}\mathrm{e}^{-(\sqrt{2}-2\sigma)(\frac{A}{2})^\gamma+(\sqrt{2}-2\sigma)B}\tilde{C}\left(\mathbb{1}_{\{B<t/3\}}2(B^{1/\gamma}\wedge r)+t\mathbb{1}_{\{B\ge t/3\}}\right)\\&\quad\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{A/2}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right)\mathrm{d}q\\&\le\frac{A}{2}\mathrm{e}^{-(\sqrt{2}-2\sigma)(\frac{A}{2})^\gamma+(\sqrt{2}-2\sigma)B}\tilde{C}\left(\mathbb{1}_{\{B<t^\gamma/3\}}2(B^{1/\gamma}\wedge r)+t\mathbb{1}_{\{B\ge t^\gamma/3\}}\right)\\&\quad\times\left(\mathbb{1}_{\{B<\frac{A}{2}\}}\mathrm{e}^{-\frac{((1-\sqrt{2}\sigma)A)^2}{A/2}}(1+o(1))+\mathbb{1}_{\{B\ge\frac{A}{2}\}}\right),\quad t\uparrow\infty.\end{aligned} \tag{3.43}$$

Using (3.43), together with the fact that, for all $t>\frac{3A^\gamma}{2}$, it holds that $\frac{t^\gamma}{3}>\frac{A}{2}$, we get that, for all such t, the sum $K\sum_{B>1}$(I3) is bounded from above by

$$\begin{aligned}&K\tilde{C}\frac{A}{2}\mathrm{e}^{-(\sqrt{2}-2\sigma)\left(\frac{A}{2}\right)^\gamma}\Big(\sum_{B>1}^{A/2}\mathrm{e}^{(\sqrt{2}-2\sigma)B}\mathrm{e}^{-\frac{2((1-\sqrt{2}\sigma)A)^2}{A}}(B^{1/\gamma}\vee r)\\&+\sum_{B>A/2}^{t^\gamma/3}\mathrm{e}^{(\sqrt{2}-2\sigma)B}(B^{1/\gamma}\vee r)+\sum_{B>t^\gamma/3}t\mathrm{e}^{(\sqrt{2}-2\sigma)B}\Big)(1+o(1)),\quad t\uparrow\infty.\end{aligned} \tag{3.44}$$

Hence, there exist t_2 and A_2 such that for all $t > t_2$ and $A > A_2$ the term in (3.44) is not greater than $\frac{\epsilon}{6}$. Now, combining the bounds in (3.37), (3.41) and (3.44), we get that, for all $t > \max\{t_0, t_1, t_2\}$ and $A > \max\{A_1, A_2\}$, $\sum_{B\geq 1} T(B,t) \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}$. By (3.4), this concludes the proof of Proposition 2.1. □

A Martingale convergence

For $\beta = \sigma + i\tau$, set $M_\beta(t) := \mathrm{e}^{-t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}+i\rho\tau\right)} \sum_{k=1}^{n(t)} \mathrm{e}^{\sigma x_k(t)+i\tau y_k(t)}$.

Proposition A.1. *For $\beta \in \mathbb{C}$ with $|\beta| < 1$, $M_\beta(t)$ is an L^2-bounded martingale with expectation one. In particular, $M_\beta(t)$ converges to a non-degenerate limit M_β a.s. and in L^2 as t tends to infinity.*

Proof. Using Representation (1.12), one easily verifies that $\mathbb{E}[M_\beta(t)] = 1$ and that it is indeed a martingale. It remains to show the L^2-boundedness of $M_\beta(t)$. We have

$$\mathbb{E}\left[|M_\beta(t)|^2\right] = \mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)}\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\sigma(x_k(t)+x_l(t))+i\tau(y_k(t)-y_l(t))}\right]. \tag{A.1}$$

Using Representation (1.12), we rewrite the right-hand side of (A.1) as

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)}\mathbb{E}\left[\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda}x_l(t)+\lambda x_k(t))+i\tau(1-\rho^2)(z_k(t)-z_l(t))}\right], \tag{A.2}$$

where $\lambda = \sigma + i\rho\tau$ and $(z_k(t))_{k\leq n(t)}$ are the particles of a BBM on $\mathbb{T}_t$ that is independent from $X(t)$. By conditioning on $\mathcal{F}^{\mathbb{T}_t}$ as in (3.19), we have that (A.2) is equal to

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)}\mathbb{E}\left[\mathrm{e}^{-(1-\rho^2)\tau^2(t-d(x_k(t),x_l(t)))}\sum_{k,l=1}^{n(t)} \mathrm{e}^{\bar{\lambda}x_l(t)+\lambda x_k(t)}\right]. \tag{A.3}$$

Similarly to (3.26), the expectation in (A.3) is equal to

$$\begin{aligned} &K\int_0^t \mathrm{d}q \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{2\pi q}}\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{\sqrt{2\pi(t-q)}} \\ &\quad\times\int_{-\infty}^{\infty}\frac{\mathrm{d}y'}{\sqrt{2\pi(t-q)}}\mathrm{e}^{2\sigma x+\sigma(y+y')+i\tau\rho(y-y')}\mathrm{e}^{-\frac{y^2+y'^2}{2}}\mathrm{e}^{-x^2/2}. \end{aligned} \tag{A.4}$$

Computing first the integrals with respect to y and y', we get that (A.4) is equal to

$$\begin{aligned} &K\int_0^t \mathrm{d}q \mathrm{e}^{2t-q-(1-\rho^2)\tau^2(t-q)+(\sigma^2-\rho^2\tau^2)(t-q)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{2\pi q}}\mathrm{e}^{2\sigma x}\mathrm{e}^{-x^2/2} \\ &= K\int_0^t \mathrm{d}q \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)}\mathrm{e}^{2\sigma^2 q}. \end{aligned} \tag{A.5}$$

Plugging (A.5) back into (A.3), we get that (A.3) is equal to

$$\mathrm{e}^{-2t\left(1+\frac{\sigma^2}{2}-\frac{\tau^2}{2}\right)}K\int_0^t \mathrm{d}q \mathrm{e}^{2t-q-\tau^2(t-q)+\sigma^2(t-q)}\mathrm{e}^{2\sigma^2 q} = K\int_0^t \mathrm{d}q \mathrm{e}^{q(\sigma^2+\tau^2-1)} \leq C, \tag{A.6}$$

for some constant $C > 0$ uniformly in t since $\sigma^2 + \tau^2 < 1$ by assumption. Hence, $M_\beta(t)$ is an L^2-bounded martingale with expectation one and converges as $t \uparrow \infty$ to a non-degenerate limit a.s. and in L^2.

□

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CHAPTER 6

Ageing at the Critical Temperature in the Random Energy Model

AGEING AT THE CRITICAL TEMPERATURE IN THE RANDOM ENERGY MODEL

ABSTRACT. We precisely quantify the asymptotic decay of a two-point correlation function in the random energy model at the critical temperature. We derive general criteria for the convergence of the clock process at the critical temperature, extending results in [10, 9]. We use these criteria to established the convergence of the properly normalized (and centered) clock process to a one stable Lévy process in the random energy model. Moreover. we establish the convergence of the centering term.

1. INTRODUCTION.

During the last decades there has been a growing interest in understanding the *ageing phenomenon* in spin glass models , see e. g. [5] for a review. The dynamics of such models show a slow relaxation to equilibrium, which is measured by the behaviour of certain two-point correlation functions. These correlation functions relate the state of the system at some time t_0 with the one at time $t_0 + t_w$.

An interesting set of models are Glauber dynamics on hypercube, namely on the state space $\Sigma_n = \{-1, 1\}^n$, that are reversible with respect to Gibbs measures associated to random Hamiltonians. In general these random Hamiltonians are given by correlated Gaussian processes indexed by Σ_n.

Most known results or predictions are based on simplified models called *trap models* and a rigorous analysis of many variants of this model was carried out over the last years (see e. g. [3, 2]).

A crucial step in analysing the behaviour of such correlation functions is to establish the convergence of the associated *clock process*. In [10] and [9] new techniques were established to ensure this convergence, namely the conditions of Durrett and Resnick [7] are applied to get convergence to certain Lévy processes. Crucial objects appearing here are variables, that are in the domain of attraction of an α-stable law. If we consider the case $\beta > \beta_c$ values for α between 0 and 1 play an important role. Thus it is natural, in the case, where the temperature is equal to the critical one, that variables in the domain of attraction of a 1-stable distribution appear. These technique has been further extended in [6] to the p-spin SK-model in the low temperature regime.

In the present paper we analyse the behaviour of the *random energy model (REM)* when it is at the critical temperature. After extending the conditions given in [10] and [9] to establish the convergence of the clock process, we are interested in the analysis of correlation functions (and thus the analysis of the ageing behaviour). The question is the following: We let the system develop and look at it at time t_0 and observe the state of the system. What is the probability that it is still in the same state at time $t_0 + t_w$? We prove that in a certain class of rescaling decays like $1/\sqrt{n}$ to zero. This is made precise in Theorem 1.3. The clock process appearing in this context is a sum of dependent random variables. Considering instead a sum of i.i.d. random variables, which are in the domain of attraction of a 1-stable law, Erikson gets in [8] a result using renewal theory. His approach relies on the i.i.d. assumpution and the precise distribution of the random variables. Moreover,

This chapter contains original results that were obtained in collaboration with Véronique Gayrard.

Bertin and Bouchaud in [4] propose the asymptotic behaviour of the correlation function for $n \to \infty$ and t, t_w large.

Before we give a detailed description of our results let us describe the setting.

1.1. **The setting.** We now specify the model Denote by $\mathcal{V}_n = \{-1,1\}^n$ the n-dimensional discrete cube, and by $\mathcal{E}_n$ its edges set. On $\mathcal{V}_n$ we construct a random landscape of traps (the random environment) by assigning to each site, x, the Boltzman weight of the REM, $\tau_n(x)$. Namely, given a parameter $\beta > 0$, called the inverse temperature, and a collection $(H_n(x), x \in \mathcal{V}_n)$ of independent standard gaussian random variables, we set $\tau_n(x) = \exp(-\beta\sqrt{n}H_n(x))$. (The dependence of $\tau_n(x)$ on β will be kept implicit.) The sequence $(\tau_n(x), x \in \mathcal{V}_n)$, $n \geq 1$, is defined on a common probability space denoted $(\Omega^\tau, \mathcal{F}^\tau, \mathbb{P})$.

Let π_n be the uniform measure on $\mathcal{V}_n$ given by

$$\pi_n(x) = 2^{-n}, \quad x \in \mathcal{V}_n. \tag{1.1}$$

The RHT dynamics in the landscape $(\tau_n(x), x \in \mathcal{V}_n)$ is a continuous time Markov chain $(X_n(t), t > 0)$ on $\mathcal{V}_n$ that can be constructed as follows: let $(J_n(k), k \in \mathbb{N})$ be the simple random walk on $\mathcal{V}_n$ with initial distribution π_n and transition probabilities

$$p_n(x,y) = \frac{1}{n}, \quad \forall (x,y) \in \mathcal{E}_n, \tag{1.2}$$

and let the *clock process* be the partial sum process

$$\widetilde{S}_n(k) = \sum_{i=0}^{k} \tau_n(J_n(i))e_{n,i}, \quad k \in \mathbb{N}, \tag{1.3}$$

where $(e_{n,i}, n \in \mathbb{N}, i \in \mathbb{N})$ is a family of independent mean one exponential random variables, independent of J_n; then

$$X_n(t) = J_n(i) \quad \text{if} \quad \widetilde{S}_n(i) \leq t < \widetilde{S}_n(i+1) \quad \text{for some} \quad i. \tag{1.4}$$

This defines X_n in terms of its jump chain, J_n, and holding times, $\tau_n(x)$ being the mean value of the exponential holding time at x. Equivalently, X_n is the chain with initial distribution μ_n and jump rates $\lambda_n(x,y) = (n\tau_n(x))^{-1}$, $(x,y) \in \mathcal{E}_n$, $x \neq y$. This last description makes it easy to check that X_n is a Glauber dynamics, namely, that it is reversible with respect to the measure (the Gibbs measure of the REM) defined on $\mathcal{V}_n$ by

$$\mathcal{G}_n(x) = \frac{\tau_n(x)}{\sum_{x \in \mathcal{V}_n} \tau_n(x)}, \quad x \in \mathcal{V}_n. \tag{1.5}$$

The model we referred to as the REM-like trap model was proposed by Bouchaud as an approximation of the ageing dynamics of the REM (see [1] for details on this derivation). It is a Markov chain X'_n on $\mathcal{V}'_n = \{1, \ldots, n\}$ with jump rates $\lambda'_n(x,y) = (n\tau'(x))^{-1}$, $(x,y) \in \mathcal{V}'_n \times \mathcal{V}'_n$, $x \neq y$, where $(\tau'(x), x \in \mathcal{V}'_n)$ are i.i.d. r.v. in the domain of attraction of a positive stable law with index $0 < \alpha < 1$

For future reference, we refer to the σ-algebra generated by the variables J_n and X_n as $\mathcal{F}^J$ and $\mathcal{F}^X$, respectively. We write P_{μ_n} for the law of the process J_n, conditional on the σ-algebra $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}_{\mu_n}$ the law of X_n conditional on $\mathcal{F}$. Observe that π_n is the invariant measure of the jump chain.

1.2. **Convergence of clock processes.** Given sequences c_n and a_n define the re-scaled clock process

$$S_n(t) = c_n^{-1}\widetilde{S}_n(\lfloor a_n t \rfloor)\,, \quad t \geq 0\,. \tag{1.6}$$

We see that c_n is the *time scale* on which we observe the process and a_n an *auxiliary time scale* that records time for the jump chain. We will distinguish two types of space scales: the *intermediate* scales and the *extreme* scales. Given a_n, let c_n be defined through

$$a_n \mathbb{P}(\tau_n(x) \geq c_n) = 1, \tag{1.7}$$

and set $m_n = \log a_n / \log 2$.

Definition 1.1. We say that a diverging sequence c_n is an *intermediate space scale* if there exists $0 < \varepsilon \leq 1$ such that

$$\frac{m_n}{n} \sim \varepsilon \quad \text{and} \quad \frac{2^{m_n}}{2^n} = o(1)\,, \tag{1.8}$$

For any of the above space scale set $\varepsilon = \lim_{n\to\infty} \frac{m_n}{n}$. Thus $0 < \varepsilon \leq 1$ if c_n is an intermediate space scale and $\varepsilon = 1$ if c_n is an extreme space scale. For $0 < \varepsilon \leq 1$ and $0 < \beta < \infty$, define

$$\begin{aligned} \beta_c(\varepsilon) &= \sqrt{\varepsilon 2 \log 2}\,, \\ \alpha(\varepsilon) &= \beta_c(\varepsilon)/\beta\,, \end{aligned} \tag{1.9}$$

and write $\beta_c \equiv \beta_c(1)$ and $\alpha \equiv \alpha(1)$. We introduce the re-scaled landscape variables by

$$\gamma_n(x) = c_n^{-1}\tau_n(x)\,, \quad x \in \mathcal{V}_n\,. \tag{1.10}$$

We now state the results on the clock process. Let us denote by $\Rightarrow$ weak convergence in the càdlàg space $D([0,\infty))$ equipped with the Skorohod J_1-topology. Define the measure ν^{int} on $(0,\infty)$ through

$$\nu^{int}(u,\infty) = u^{-\alpha(\varepsilon)}\alpha(\varepsilon)\Gamma(\alpha(\varepsilon))\,, \quad u > 0\,. \tag{1.11}$$

Theorem 1.2. *For all $0 < \varepsilon \leq 1$ and all $0 < \beta < \infty$ such that $\alpha(\varepsilon) = 1$,*

$$S_n - M_n \Rightarrow S^{crit}\,, \tag{1.12}$$

where S^{crit} is the Lévy process with Lévy triple $(0, 0, \nu^{int})$ and

$$M_n(t) = \sum_{i=1}^{[a_n t]} \sum_{x\in\mathcal{V}_n} p_n\left(J_n(i-1), x\right)\gamma_n(x)\left(1 - e^{1/\gamma_n(x)}\right)\,. \tag{1.13}$$

Moreover for all $T > 0$ and all $\epsilon > 0$ we have

$$\lim_{n\to\infty} \mathcal{P}\left(\sup_{t\in[0,T]} \left| M_n(t) - \mathbb{E}(E(M_n(1)))t \right| > \epsilon\right) = 0\,. \tag{1.14}$$

In the case $\alpha(\varepsilon) = 1$ S^{crit} is not a subordinator but a compensated pure jump Lévy process.

1.3. **Consequences for correlation functions.** To study the ageing behavior of X_n described by (1.4) we need to choose three ingredients: 1) an initial distribution, μ_n; 2) a time scale of observation, c_n; and 3) a time-time correlation function, $\mathcal{C}_n(t,s)$, $t,s \geq 0$: this is a function that quantifies the correlation between the state of the system at time t, $X_n(c_n t)$, and its state at time $t+s$, $X_n(c_n(t+s))$.

A natural choice of correlation function, in view of ageing results in the REM, is

$$\mathcal{C}_n(t_n, t_n+s_n) = \mathcal{P}_{\mu_n}\left(\left\{\widetilde{S}_n(k)\,, k\in\mathbb{N}\right\}\cap(c_n t_n, c_n(t_n+s_n)) = \emptyset\right)\,, 0\leq t_n < t_n+s_n\,. \tag{1.15}$$

When $0<\alpha(\varepsilon)<1$ one chooses $t_n = t$ and $s_n = \rho t$ with $t,\rho>0$. We will say that the process X_n has an arcsine ageing regime of parameter $0<\alpha<1$ whenever one can find a time-time correlation function such that, denoting by $\mathrm{Asl}_\alpha(\cdot)$ the generalized arcsine distribution function of parameter α,

$$\mathrm{Asl}_\alpha(u) = \frac{\sin\alpha\pi}{\pi}\int_0^u (1-x)^{-\alpha}x^{-1+\alpha}dx\,,\quad 0<\alpha<1\,, \tag{1.16}$$

one of the following relations holds true,

$$\begin{aligned}
\lim_{t\to 0}\lim_{n\to\infty}\mathcal{C}_n(t,t+\rho t) &= \mathrm{Asl}_\alpha(1/1+\rho)\,, && (1.17)\\
\lim_{n\to\infty}\mathcal{C}_n(t,t+\rho t) &= \mathrm{Asl}_\alpha(1/1+\rho)\,,\ \ t>0 \ \text{arbitrary}, && (1.18)\\
\lim_{t\to\infty}\lim_{n\to\infty}\mathcal{C}_n(t,t+\rho t) &= \mathrm{Asl}_\alpha(1/1+\rho) && (1.19)
\end{aligned}$$

for all $\rho\geq 0$, and some convergence mode w.r.t. the probability law $\mathbb{P}$ of the random landscape. It is today well understood that the existence of an arcsine ageing regime is governed by Dynkin and Lamperti's arcsine law for subordinators, applied to the limiting, appropriately re-scaled, clock process: arcsine ageing will be present when the re-scaled clock process converges to a subordinator whose Lévy measure satisfies the slow variation conditions of the Dynkin-Lamperti Theorem[1].

In the next theorem we state the behaviour of the correlation function in a subregion of the intermediate scales with $\beta_c = \beta$. Namely, we look at all scales such that

$$\lim_{n\to\infty}\frac{\log c_n}{\beta\sqrt{n}} - \beta\sqrt{n} = \theta \tag{1.20}$$

for some constant $\theta\in(-\infty,\infty)$. We choose the waiting time t_n to be of the order of the centering term, namely $t_n = \mathbb{E}(E(M_n(1)))$. Due to our additional assumption on c_n we know that $\mathbb{E}(E(M_n(1))) = \beta\sqrt{2\pi n}\Phi(\theta)(1+o(1))$. Moreover, in the critical temperature case the sub-leading orders of c_n and the ratio between a_n and c_n and the behaviour of $\mathbb{E}(E(M_n(1)))$ play a crucial role for the shape of the limiting function we observe.

Theorem 1.3. *Let $\beta=\beta_c(\varepsilon)$ with $0<\varepsilon\leq 1$. Let c_n be an intermediate scale with $\lim_{n\to\infty}\frac{\log c_n}{\beta\sqrt{n}} - \beta\sqrt{n} = \theta$ for some constant $\theta\in(-\infty,\infty)$. Let $M_n(1)$ be defined as in (1.13). Set*

$$\begin{aligned}
t_n &= c_n t,\\
s_n &= c_n s. && (1.21)
\end{aligned}$$

[1]See e.g. Appendix A.2.1 of [10] for a statement of the latter.

for some $s \in \mathbb{R}_{>0}$. *Then we have* $\mathbb{P}$*-a.s if* $\sum \frac{a_n}{2^n} < \infty$ *and in* $\mathbb{P}$*-probability otherwise*

$$\lim_{n\to\infty} \sqrt{n}\mathcal{C}_n(t_n, t_n + s_n) = \frac{e^{-\theta^2/2}}{\Phi(\theta)\beta\sqrt{2\pi}} \log\left(1 + \frac{1}{s}\right). \tag{1.22}$$

The remainder of the paper is organized as follows. In the next section we implement sufficient conditions for the convergence to a pure jump Lévy process. Moreover we give sufficient conditions for (1.14) to hold.

2. Key tools and strategy.

Recall that the initial distribution is given by π_n. We now formulate conditions for the sequence S_n to converge. The idea of proof is taken from Theorem 1.1 of [10]. We state these conditions for given sequences c_n and a_n, and for a fixed realization of the random landscape, i.e. for fixed $\omega \in \Omega^\tau$, and do not make this explicit in the notation. For $u \in (0, \infty)$ and $\delta > 0$ set

$$f_\delta(u) \;=\; u^2(1 - e^{-\delta/u}) - \delta u e^{-\delta/u}. \tag{2.1}$$

Condition (A0). For all $u > 0$

$$2^{-n} \sum_{x\in\mathcal{V}_n} e^{-u/\gamma_n(x)} = o(1). \tag{2.2}$$

Condition (A1). There exists a σ-finite measure ν on $(0, \infty)$ such that $\nu(u, \infty)$ is continuous, and such that, for all $t > 0$ and all $u > 0$,

$$P\left(\left|\sum_{j=1}^{\lfloor a_n t\rfloor} \sum_{x\in\mathcal{V}_n} p_n(J_n(j-1), x)e^{-u/\gamma_n(x)} - t\nu(u, \infty)\right| < \epsilon\right) = 1 - o(1), \quad \forall\epsilon > 0. \tag{2.3}$$

Condition (A2). For all $u > 0$ and all $t > 0$,

$$P\left(\sum_{j=1}^{\lfloor a_n t\rfloor} \Big[\sum_{x\in\mathcal{V}_n} p_n(J_n(j-1), x)e^{-u/\gamma_n(x)}\Big]^2 < \epsilon\right) = 1 - o(1), \quad \forall\epsilon > 0. \tag{2.4}$$

Condition (A3'). For all $u > 0$ and all $t > 0$,

$$\lim_{\delta\to 0}\lim_{n\to\infty} \frac{[a_n t]}{2^n} \sum_{x\in\mathcal{V}_n} f_\delta(\gamma_n(x)) = 0. \tag{2.5}$$

Theorem 2.1. *For all sequences* a_n *and* c_n *for which Conditions (A0), (A1) and (A2) are satisfied* $\mathbb{P}$*-almost surely respectively in* $\mathbb{P}$*-probability, the following holds: If* $\nu(du) = u^{-2}du$ *and Condition (A3') are verified w.r.t. the same convergence mode as Condition (A1), then :*

$$S_n - M_n \Rightarrow S^{crit}, \tag{2.6}$$

$\mathbb{P}$*-almost surely, respectively in* $\mathbb{P}$*-probability, and where* S^{crit} *is a Lévy process with Lévy triple* $(0, 0, \nu)$.

Proof. Let us first work on a fixed realization of the landscape. Conditions (A1) and (A2) are those of Theorem 1.1 of [10] when the initial distribution is the invariant measure π_n. Moreover in this case Condition (A0) is Condition (A0') of Theorem 1.1 in [10]. To see that in case (ii) Condition (A3') implies Condition (d) of Theorem 4.1 of [7], which then implies Condition (c), we have to consider

$$\overline{Z}^\delta_{n,i} = Z_{n,i}\mathbb{1}_{\{Z_{n,i}\le\delta\}} - \mathcal{E}\left(Z_{n,i}\mathbb{1}_{\{Z_{n,i}\le\delta\}} \mid \mathcal{F}_{n,i-1}\right), \tag{2.7}$$

where $\mathcal{F}_{n,i-1} = \sigma\left(e_{n,1}, \ldots, e_{n,i-1}, J_n(1), \ldots, J_n(i-1)\right)$ and show that

$$\lim_{\delta\to 0}\lim_{n\to\infty} \mathcal{P}\Big(\sum_{i=1}^{[a_n t]} \mathcal{E}\Big(\big(\overline{Z}_{n,i}^{\delta}\big)^2 \Big| \mathcal{F}_{n,i-1} \Big) > \epsilon \Big) = 0\,. \tag{2.8}$$

By a first order Tchebychev inequality the probability in (2.8) is bounded above by

$$\epsilon^{-1} \sum_{i=1}^{[a_n t]} \mathcal{E}\left(Z_{n,i} \mathbb{1}_{\{Z_{n,i}\leq\delta\}} \right)^2 = 2^{-n} \sum_{x\in\mathcal{V}_n} f_\delta(\gamma_n(x))\,. \tag{2.9}$$

To apply the proof of Theorem 4.1 of [7] we have to additionally verify that fom (A1)follows that as $n\to\infty$

$$\sum_{i=1}^{[a_n t]} \mathcal{E}_{\pi_n}\left(Z_{n,i} \mathbb{1}_{\{\delta < Z_{n,i} < \gamma\}} \,|\mathcal{F}_{n,i-1} \right) \to t \int_\delta^1 x\nu(\mathrm{d}x) \quad \text{in } \mathcal{P}\text{-probability}\,. \tag{2.10}$$

This can be shown as proposed in the proof of Theorem 4.1 in [7] using a Riemannsum argument. Let $k\in\mathbb{N}$. Taking an equidistant partition $t_0,\ldots,t_k$ of $[\delta, 1]$ one can bound $Z_{n,i}$ in the following way:

$$\sum_{j=0}^{k-1} t_j \mathbb{1}_{\{t_j\leq Z_{n,i} < t_{j+1}\}} \leq Z_{n,i} \leq \sum_{j=0}^{k-1} t_{j+1} \mathbb{1}_{\{t_j\leq Z_{n,i} < t_{j+1}\}}\,. \tag{2.11}$$

We now take conditional expectations w.r.t. $\mathcal{F}_{n,i-1}$ and use Condition (A1). So far we kept a realization of the random landscape fix. Arguing as in the proof of Theorem 1.1 in [10] we conclude that if Conditions (A0), (A1), (A2) and (A3') are satisfied $\mathbb{P}$-a.s., we have that (2.6) is satisfied $\mathbb{P}$-a.s., and that if in case (i) Conditions (A0), (A1), (A2) and (A3') are satisfied in $\mathbb{P}$-probability, we have that (2.6) is satisfied in $\mathbb{P}$-probability.

□

3. Properties of the Landscape

In this section we review the needed properties of the re-scaled landscape variables $(\gamma_n(x), x\in\mathcal{V}_n)$, and most importantly, the heavy tailed nature of their distribution, obtained in Lemma 2.1 in [9].

We assume that $0<\beta<\infty$ is fixed, and as before, drop all dependence on β in the notation. Since this is a continuous monotone decreasing function, it has a well defined inverse $G_n^{-1}(u) := \inf\{y\geq 0 : G_n(y)\leq u\}$. For $v\geq 0$ set

$$h_n(v) = a_n \mathbb{P}(\tau_n(x) > c_n v)\,. \tag{3.1}$$

Lemma 3.1. *Let c_n be any of the space scales of Definition 1.1.*
(i) For each fixed $\zeta>0$ and all n sufficiently large so that $\zeta > c_n^{-1}$, the following holds: for all v such that $\zeta\leq v<\infty$,

$$h_n(v) = v^{-\alpha_n}(1+o(1))\,, \tag{3.2}$$

where $0\leq\alpha_n = \alpha(\varepsilon)+o(1)$.
(ii) Let $0<\delta<1$. Then, for all v such that $c_n^{-\delta}\leq v\leq 1$ and all large enough n,

$$v^{-\alpha_n}(1+o(1)) \leq h_n(v) \leq \tfrac{1}{1-\delta} v^{-\alpha_n(1-\frac{\delta}{2})}(1+o(1))\,, \tag{3.3}$$

where α_n is as before.

More precisely, inspecting the proof of Lemma 2.1 in [9] one obtains for $\beta = \beta_c$ the following asymptotics for a_n, c_n and α_n.

$$\begin{aligned} \log a_n &= \tfrac{1}{2}\beta^2(\varepsilon)n(1+o(1))\,, && (3.4)\\ \frac{\log c_n}{\beta\sqrt{n}} &= (2\log a_n)^{\frac12} - \tfrac12(\log\log a_n + \log 4\pi)/(2\log a_n)^{\frac12} + \mathcal{O}(1/\log a_n), && (3.5)\\ \alpha_n &= (\sqrt{n}\beta)^{-1}B_n = \alpha(\varepsilon)(1+o(1))\,. \end{aligned}$$

This will be a key property on which the rest of this paper relies heavily.

4. Control of the centring term $M_n(t)$

Theorem 4.1. *For all intermediate scales c_n with $\beta = \beta_c$ the following holds.*

(i) *If $\sum \frac{a_n}{2^n} < \infty$ then*

$$M_n(t) - \mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right) \tag{4.1}$$

converges $\mathbb{P}$-a.s in $\mathcal{P}$-probability to zero.

(ii) *If $\sum \frac{a_n}{2^n} = \infty$ then*

$$M_n(t) - \mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right) \tag{4.2}$$

converges in $\mathbb{P}$-probability and in $\mathcal{P}$-probability to zero.

We start by calculating its mean value with respect to the landscape and then show concentration in Lemma 4.5. A simple calculation yields

$$\mathcal{E}\left(M_n(t)\right) = \frac{[a_n t]}{2^n}\sum_{x\in\mathcal{V}_n}\gamma_n(x)\left(1-e^{-\gamma_n(x)^{-1}}\right). \tag{4.3}$$

To simplify the representation we use the notation

$$g(u) = u(1-e^{-u^{-1}}), \quad u>0. \tag{4.4}$$

To study the behavior of $M_n(t)$ we need a control on the moments of $g(\gamma_n(x))$ which is done in the following Lemma A.2 in the appendix. Moreover we fix the notation

$$G_n(y) = \sum_{x\in\mathcal{V}_n} p_n(x,y)\gamma_n(x)\left(1-e^{-\gamma_n(x)^{-1}}\right) = \sum_{x\in\mathcal{V}_n} p_n(x,y)g(\gamma_n(x)). \tag{4.5}$$

Proposition 4.2. *Let ρ_n be a decreasing sequence $\rho_n \to 0$ as $n\to\infty$. Then there exists a sequence $\Omega^\tau_{n,0} \subset \Omega^\tau$ with $\mathbb{P}\left(\left(\Omega^\tau_{n,0}\right)^c\right) \le \frac{\theta_n}{a_n\rho_n}$ such that on $\Omega^\tau_{n,0}$*

$$P\left(|M_n(t) - \mathcal{E}\left(M_n(t)\right)| \ge \epsilon\right) \le \epsilon^{-2}\left((a)+(b)+(c)+(d)\right), \tag{4.6}$$

where

$$\begin{aligned} (a) &= \frac{[a_n t]^2}{2^n}\left(\sum_{x\in\mathcal{V}_n} G_n(x)\pi_n(x)\right)^2 && (4.7)\\ (b) &= \frac{[a_n t]}{2^n}\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n} p_n^2(x,x')g(\gamma_n(x))g(\gamma_n(x')) && (4.8)\\ (c) &= cn^{-2}\sum_{z\in\mathcal{V}_n}[a_n t]\pi_n(z)g(\gamma_n(z))^2 && (4.9)\\ (d) &= \rho_n\left(\mathbb{E}\left([a_n t]\sum_{x\in\mathcal{V}_n} G_n(x)\pi_n(x)\right)\right)^2 && (4.10) \end{aligned}$$

Proof. We do a second order Tchebychev inequality and then we use the same bounds on the jump chain as in the Verification of (A1) and (A2). Thus we obtain

$$
\begin{aligned}
P\left(\left|\frac{M_n(t)}{c_n}-\mathcal{E}\left(\frac{M_n(t)}{c_n}\right)\right|\geq\epsilon\right) &\leq \epsilon^{-2}E_{\pi_n}\left([a_nt]|\sum_{y\in\mathcal{V}_n}\left(\pi_n^{J,t}(y)-\pi_n(y)\right)G_n(y)\right)^2 \\
&\leq \epsilon^{-2}\left((I)+(II)+(III)\right), \qquad (4.11)
\end{aligned}
$$

where $(I),(II)$, and (III) are the corresponding terms to the ergodic theorem of Proposition 4.1 in citeG2 when replacing $h_n^u(x)$ by $G_n(u)$. We control these terms in the same way as in the Verification of (A2). For (I) we get

$$
(I)\leq\tfrac{[a_nt]^2}{2^n}\left(\textstyle\sum_{x\in\mathcal{V}_n}G_n(x)\pi_n(x)\right)^2. \qquad (4.12)
$$

For (II) we have

$$
(II)\leq\frac{[a_nt]}{2^n}\sum_{x\in\mathcal{V}_n}\sum_{x'\in\mathcal{V}_n}p_n^2(x,x')g(\gamma_n(x))g(\gamma_n(x')). \qquad (4.13)
$$

With (III) we proceed as in the proof of Proposition 4.1 in [9] and get

$$
\begin{aligned}
(III) &= 2[a_nt]\sum_{l=1}^{\theta_n-1}\sum_{z\in\mathcal{V}_n}\Big(\sum_{x\in\mathcal{V}_n}\pi_n(x)G_n(x)p_n^{l+1}(x,z)\Big)g(\gamma_n(z)) \qquad (4.14)\\
&\equiv 2\sum_{l=1}^{\theta_n-1}\left((III)_{1,l}+(III)_{2,l}\right), \qquad (4.15)
\end{aligned}
$$

where $(III)_{1,l}$ contains the terms with $x=z$ and $(III)_{2,l}$ those with $x\neq z$.

$$
\begin{aligned}
(III)_{1,l} &= \sum_{z\in\mathcal{V}_n}[a_nt]\pi_n(z)g(\gamma_n(z))^2p_n^{l+2}(z,z). \qquad (4.16)\\
(III)_{2,l} &= \sum_{z,y\in\mathcal{V}_n}[a_nt]\pi_n(y)g(\gamma_n(z))g(\gamma_n(y))p_n^{l+2}(y,z). \qquad (4.17)
\end{aligned}
$$

For $(III)_{1,l}$ we have by Proposition 3.1 of [9]

$$
\sum_{l=1}^{\theta_n}(III)_{1,l}\leq cn^{-2}\sum_{z\in\mathcal{V}_n}[a_nt]\pi_n(z)g(\gamma_n(z))^2 \qquad (4.18)
$$

For the term $(III)_{2,l}$ we use the following Lemma.

Lemma 4.3. *Let $\rho_n\to 0$ as $n\to\infty$. Then there exists a sequence $\Omega^\tau_{n,0}\subset\Omega^\tau$ s.t $\mathbb{P}\left(\left(\Omega^\tau_{n,0}\right)^c\right)<\frac{\theta_n}{\rho_na_n}$ and s.t. on $\Omega^\tau_{n,0}$*

$$
\sum_{l=1}^{\theta_n}(III)_{2,l}<\rho_n\left(\mathbb{E}\left([a_nt]\sum_{x\in\mathcal{V}_n}G_n(x)\pi_n(x)\right)\right)^2 \qquad (4.19)
$$

Proof. Since $y\neq z$ we can use independence of $\gamma_n(y)$ and $\gamma_n(z)$. By a first order Tchebychev inequality we have

$$
\mathbb{P}\left(\textstyle\sum_{l=1}^{\theta_n}(III)_{2,l}\geq\eta\right)\leq\eta^{-1}\textstyle\sum_{l=1}^{\theta_n-1}\mathbb{E}(III)_{2.l} \qquad (4.20)
$$

Now we have

$$
\begin{aligned}
\sum_{l=1}^{\theta_n-1} \mathbb{E}(III)_{2.l} &\leq \frac{1}{[a_n t]}\left(\mathbb{E}\left([a_n t]\sum_{x\in\mathcal{V}_n} G_n(x)\pi_n(x)\right)\right)^2 \sum_{l=1}^{\theta_n-1}\sum_{z\in\mathcal{V}_n} p_n^l(y,z) \\
&\leq \frac{\theta_n}{[a_n t]}\left(\mathbb{E}\left([a_n t]\sum_{x\in\mathcal{V}_n} G_n(x)\pi_n(x)\right)\right)^2 \qquad (4.21)
\end{aligned}
$$

This yields $\mathbb{P}\left(\sum_{l=1}^{\theta_n}(III)_{2,l} \geq \eta\right) \leq \frac{\theta_n}{\eta[a_n t]}\left(\mathbb{E}\left([a_n t]\sum_{x\in\mathcal{V}_n} G_n(x)\pi_n(x)\right)\right)^2$. □

Collecting all the bounds finishes the proof of Lemma 4.2. □

Now we want to analyze the terms appearing in Lemma 4.2 separately. In a first step, we are interested in computing the expected value with respect to the random environment as $n \to \infty$ and then show concentration around the mean value. We start by having a look at (b).

$$
\mathbb{E}\left((b)\right) = (b_1) + (b_2), \qquad (4.22)
$$

where

$$
\begin{aligned}
(b_1) &= \frac{[a_n t]}{n}\mathbb{E}\left(g(\gamma_n(x))^2\right), \qquad (4.23) \\
(b_2) &= \frac{[a_n t]}{2^n}\sum_{x\in\mathcal{V}_n}\sum_{\substack{x'\in\mathcal{V}_n,\\ x\neq x'}} p_n^2(x,x')\mathbb{E}\left(g(\gamma_n(x))g(\gamma_n(x'))\right). \qquad (4.24)
\end{aligned}
$$

For (b_1) we know from the calculations in (A.12) that $(b_1) = C\frac{1}{n}(1+o(1))$ for some constant $C > 0$. For (b_2) we have

$$
\begin{aligned}
(b_2) &= \frac{n-1}{n}[a_n t]\left(\mathbb{E}\left(g(\gamma_n(x))\right)\right)^2 \qquad (4.25) \\
&\leq \frac{n-1}{n}[a_n t]\left(\frac{e^{\beta^2 n/2}}{c_n}\right)^2 \sim \frac{\sqrt{n}}{c_n}. \qquad (4.26)
\end{aligned}
$$

Now we have a closer look at $\mathbb{E}((a))$.

$$
\mathbb{E}\left((a)\right) = (c') + (f), \qquad (4.27)
$$

where

$$
\begin{aligned}
(f) &= \frac{[a_n t]^2}{2^n}\mathbb{E}\left(\sum_{x\in\mathcal{V}_n}\sum_{y\in\mathcal{V}_n, x\neq y}\pi_n(x)\pi_n(y)G_n(x)G_n(y)\right), \qquad (4.28) \\
(c') &= \frac{[a_n t]^2}{2^{3n}}\mathbb{E}\left(\sum_{a\in\mathcal{V}_n} G_n(x)^2\right) \qquad (4.29)
\end{aligned}
$$

We write

$$
(f) \equiv (g) + (h), \qquad (4.30)
$$

where

$$
\begin{aligned}
(g) &= \frac{[a_n t]^2}{2^3 n} \sum_{y\in\mathcal{V}_n} \sum_{x\in\mathcal{V}_n, x\neq y} \sum_{x'\in\mathcal{V}_n} p_n(y,x')p_n(x,x')\mathbb{E}\left(g(\gamma_n(x'))^2\right) \\
&\leq \frac{[a_n t]^2}{2^n}\mathbb{E}\left(g(\gamma_n(x'))^2\right) \leq c_2\frac{[a_n t]}{2^n} \quad \text{for } n \text{ large enough}, c_2>0, \qquad (4.31)\\
(h) &= \frac{[a_n t]^2}{2^{3n}} \sum_{y\in\mathcal{V}_n} \sum_{\substack{x\in\mathcal{V}_n,\\ x\neq y}} \sum_{x'\in\mathcal{V}_n} \sum_{\substack{y'\in\mathcal{V}_n,\\ x'\neq y'}} p_n(y,y')p_n(x,x')\left(\mathbb{E}g(\gamma_n(x))\right)^2 \\
&\leq \frac{[a_n t]^2 e^{\beta^2 n}}{2^n c_n^2} \leq \frac{n}{2^n} \quad \text{for } n \text{ large enough.} \qquad (4.32)
\end{aligned}
$$

Now we finally look at (c').

$$
(c') = (i) + (j), \tag{4.33}
$$

where

$$
\begin{aligned}
(i) &= \frac{[a_n t]^2}{2^{3n}} \sum_{z\in\mathcal{V}_n} \sum_{y\in\mathcal{V}_n} p_n(z,y)^2\mathbb{E}\left(g(\gamma_n(x'))^2\right) \qquad (4.34)\\
&\leq \frac{[a_n t]}{n2^{2n}}[a_n t]\mathbb{E}\left(g(\gamma_n(x')^2)\right) \leq c_2\frac{[a_n t]}{n2^{2n}} \quad \text{for } n \text{ large enough} \qquad (4.35)\\
(j) &= \frac{[a_n t]^2}{2^{3n}} \sum_{z\in\mathcal{V}_n} \sum_{\substack{y,y'\in\mathcal{V}_n,\\ y\neq y'}} p_n(z,y)p_n(z,y')\left(\mathbb{E}g(\gamma_n(y))\right)^2 \qquad (4.36)\\
&\leq \frac{[a_n t]}{2^{2n}}[a_n t]\left(\mathbb{E}g(\gamma_n(y))\right)^2 \leq c_2\frac{[a_n t]}{2^{2n}} \quad \text{for } n \text{ large enough} \qquad (4.37)
\end{aligned}
$$

Finally we calculate $\mathbb{E}((c))$ and$\mathbb{E}((d))$ using the same method as above. For n large enough this yields

$$
\mathbb{E}\left((c)\right) \leq c_2 n^{-2}, \quad \mathbb{E}\left((d)\right) \leq c_2\rho_n. \tag{4.38}
$$

Now we need the concentration of (a), (b) and (c) around there mean values with respect to the random environment.

Lemma 4.4. *(i) If* $\sum \frac{[a_n t]}{2^n} < \infty$ *then there exists* $\Omega^\tau_{a,b,c} \subset \Omega^\tau$ *with* $\mathbb{P}\left(\Omega^\tau_{a,b,c}\right) = 1$ *such that on* $\Omega^\tau_{a,b,c}$ *the following holds for all* $t>0$

$$
\begin{aligned}
\lim_{n\to\infty} |(a) - \mathbb{E}\left((a)\right)| &= 0 \\
\lim_{n\to\infty} |(b) - \mathbb{E}\left((b)\right)| &= 0 \\
\lim_{n\to\infty} |(c) - \mathbb{E}\left((c)\right)| &= 0. \qquad (4.39)
\end{aligned}
$$

(ii) If $\sum \frac{[a_n t]}{2^n} = \infty$ *there exists* $\Omega^\tau_{a,b,c,n} \subset \Omega^\tau$ *with* $\mathbb{P}\left(\left(\Omega^\tau_{a,b,c,n}\right)^c\right) = 1 - o(1)$ *such that for* n *large enough on* $\Omega^\tau_{a,b,c,n}$ *the following holds for all* $t > 0$*:*

$$\begin{aligned} |(a) - \mathbb{E}\left((a)\right)| &\leq \left(\frac{[a_n t]}{2^n}\right)^{1/4} \\ |(b) - \mathbb{E}\left((b)\right)| &\leq \left(\frac{[a_n t]}{2^n}\right)^{1/4} \\ |(c) - \mathbb{E}\left((c)\right)| &\leq \left(\frac{[a_n t]}{2^n}\right)^{1/4} \end{aligned} \tag{4.40}$$

Proof. Throughout the proof let $C > 0$ be a generic constant that is large enough to fulfil all desired inequalities. We start by having a look at (b). Using again a second order Tchebychev inequality we have

$$\mathbb{P}\left(|(b) - \mathbb{E}\left((b)\right)| > t\right) \leq t^{-2}\left(\theta_1 + \theta_2\right), \tag{4.41}$$

where

$$\theta_1 = \left(\frac{[a_n t]}{2^n}\right)^2 \sum_{y \in \mathcal{V}_n} \mathbb{E}\left(G_n(y)^2 - \mathbb{E}\left(G_n(y)\right)^2\right)^2 \tag{4.42}$$

$$\theta_2 = \left(\frac{[a_n t]}{2^n}\right)^2 \sum_{y \in \mathcal{V}_n} \sum_{\substack{y' \in \mathcal{V}_n, \\ y \neq y'}} \mathbb{E}\left(G_n(y)^2 - \mathbb{E}\left(G_n(y)\right)^2\right)\left(G_n(y')^2 - \mathbb{E}\left(G_n(y')\right)^2\right) \tag{4.43}$$

Hence we observe that the expectation with respect to the random environment of all terms appearing converges to 0 as $n \to \infty$. First we bound θ_1 by

$$\theta_1 \leq \left(\frac{[a_n t]}{2^n}\right)^2 \sum_{y \in \mathcal{V}_n} \mathbb{E}\left(G_n(y)^4\right) - \left(\mathbb{E}\left(G_n(y)\right)^2\right)^2. \tag{4.44}$$

By explicit calculation we have

$$\begin{aligned} &\left(\frac{[a_n t]}{2^n}\right)^2 \sum_{y \in \mathcal{V}_n} \mathbb{E}\left(G_n(y)^4\right) \\ \leq\ &\left(\frac{[a_n t]}{2^n}\right)^2 \Big(\sum_{y \in \mathcal{V}_n} \sum_{x \in \mathcal{V}_n} p_n(y,x)^4 \mathbb{E}\left(g(\gamma_n(x))^4\right) \\ +\ & C \sum_{y \in \mathcal{V}_n} \sum_{x \in \mathcal{V}_n} \sum_{\substack{x'' \in \mathcal{V}_n, \\ x \neq x'}} p_n(y,x) p_n(y,x')^3 \mathbb{E}\left(g(\gamma_n(x)) g(\gamma_n(x')^3)\right) \\ +\ & C \sum_{y \in \mathcal{V}_n} \sum_{x \in \mathcal{V}_n} \sum_{\substack{x' \in \mathcal{V}_n, \\ x \neq x'}} p_n(y,x)^2 p_n(y,x')^2 \mathbb{E}\left(g(\gamma_n(x))^2 g(\gamma_n(x'))^2\right) \\ +\ & C \sum_{y \in \mathcal{V}_n} \sum_{x \in \mathcal{V}_n} \sum_{\substack{x' \in \mathcal{V}_n, \\ x \neq x'}} \sum_{\substack{x'' \in \mathcal{V}_n, \\ x'' \neq x', x'' \neq x}} p_n(y,x)^2 p_n(y,x') p_n(y,x'') \mathbb{E}\left((g(\gamma_n(x))^2 g(\gamma_n(x')) g(\gamma_n(x''))\right) \\ +\ & \sum_{y \in \mathcal{V}_n} \sum_{x \in \mathcal{V}_n} \sum_{\substack{x' \in \mathcal{V}_n, \\ x \neq x'}} \sum_{\substack{x'' \in \mathcal{V}_n, \\ x'' \neq x', \\ x'' \neq x}} \sum_{\substack{x''' \in \mathcal{V}_n, \\ x''' \neq x', \\ x''' \neq x, x''' \neq x''}} \mathbb{E}(((g(\gamma_n(x)) g(\gamma_n(x')) g(\gamma_n(x'')) g(\gamma_n(x''')))\Big) \end{aligned} \tag{4.45}$$

Using the calculations in Lemma A.2 on the behavior of $g(\gamma_n(x))$ and the independence of $\gamma_n(x)$ and $\gamma_n(y)$ when $x \neq y$ we have

$$\theta_1 \leq \frac{[a_n t]}{2^n}\left(\frac{1}{n^3} + C\frac{e^{n\beta^2/2}}{n^2 c_n} + \frac{C}{n^2 2^n [a_n t]} + C\frac{e^{n\beta^2}}{n c_n^2} + \frac{e^{2n\beta^2}[a_n t]}{c_n^4}\right) \tag{4.46}$$

Expanding the expression θ_2 we obtain

$$\theta_2 \left(\frac{[a_n t]}{2^n}\right)^2 \sum_{y\in\mathcal{V}_n} \sum_{\substack{y'\in\mathcal{V}_n,\\ y\neq y'}} \mathbb{E}\left(G_n(y)^2 G_n(y')^2\right) - \mathbb{E}\left(G_n(y)^2\right)\mathbb{E}\left(G_n(y')^2\right). \tag{4.47}$$

We observe that when we know expand the terms involved in this expression certain terms drop. Keeping the remaining terms we bound θ_2 by

$$\begin{aligned}
\theta_2 \;\leq\; & \left(\frac{[a_n t]}{2^n}\right)^2 \Big(\sum_{y\in\mathcal{V}_n} \sum_{\substack{y'\in\mathcal{V}_n,\\ y\neq y'}} \sum_{x\in\mathcal{V}_n} p_n(y,x)^2 p_n(y',x)^2 \mathbb{E}\left(g(\gamma_n(x))^4\right) \\
+\; & C \sum_{y\in\mathcal{V}_n} \sum_{\substack{y'\in\mathcal{V}_n,\\ y\neq y'}} \sum_{x\in\mathcal{V}_n} \sum_{\substack{x'\in\mathcal{V}_n,\\ x'\neq x}} p_n(y,x)p_n(y,x')p_n(y',x)^2 \mathbb{E}\left(g(\gamma_n(x))^3 g(\gamma_n(x'))\right) \\
+\; & C \sum_{y\in\mathcal{V}_n} \sum_{\substack{y'\in\mathcal{V}_n,\\ y\neq y'}} \sum_{x\in\mathcal{V}_n} \sum_{\substack{x'\in\mathcal{V}_n,\\ x'\neq x}} \sum_{\substack{z'\in\mathcal{V}_n,\\ z'\neq x, z'\neq x'}} p_n(y,x)p_n(y,x')p_n(y',x)p_n(y',z') \\
& \mathbb{E}\left(g(\gamma_n(x))^2 g(\gamma_n(x'))g(\gamma_n(z'))\right)\Big)
\end{aligned} \tag{4.48}$$

Now we use the calculations of Lemma A.2 on $g(\gamma_n(x))$ to obtain the asymptotic behavior of θ_2.

$$\theta_2 \;\leq\; \frac{[a_n t]}{2^n}\left(\frac{1}{n^2} + C\frac{e^{n\beta^2/2}}{n c_n} + C\frac{e^{n\beta^2}}{c_n^2}\right) \tag{4.49}$$

Collecting the bounds gives

$$\mathbb{P}\left(|(b) - \mathbb{E}\left((b)\right)| > t\right) \leq Ct^{-2}\frac{[a_n t]}{2^n}\left(\frac{e^{2n\beta^2}[a_n t]}{c_n^4} + \frac{1}{n^2} + \frac{e^{n\beta^2/2}}{n c_n} + \frac{e^{n\beta^2}}{c_n^2}\right) \tag{4.50}$$

To show the concentration of (a) we decompose (a) in the same way as in (4.27). And show concentration for the two parts. The concentration of the first one follows in exactly the same way as for (b). Looking at the second we observe that using a second order Tchebychev inequality that we have to control

$$\mathbb{P}\left(|(f) - \mathbb{E}(f)| > \epsilon\right) \leq \epsilon^{-2}\mathbb{E}\left(((f) - \mathbb{E}((f)))^2\right. \tag{4.51}$$

Expanding this expression and sorting them according to the amount of g_n's that are equal we observe that the terms appearing are of the same shape as those in θ_2 just with a smaller prefactor which is given by $\frac{[a_n t]^4}{2^{6n}}$. By a gross estimate we have the following bound:

$$\mathbb{P}\left(|(f) - \mathbb{E}(f)| > \epsilon\right) \leq C\frac{[a_n t]^2}{2^{2n}} \tag{4.52}$$

Finally we have a look at (c). Using again a second order Tchebychev inequality

$$\mathbb{P}\left(|(c) - \mathbb{E}(c)| > \epsilon\right) \leq \epsilon^{-2}(\mathbb{E}\left(((c) - \mathbb{E}((c)))^2\right. \leq C\frac{[a_n t]}{n^4 2^n}. \tag{4.53}$$

Thus we have finally shown concentration of all Terms involved in 4.2. $\square$

The following Lemma proves the concentration of $\mathcal{E}\left(M_n(t)\right)$ around $\mathbb{E}\mathcal{E}\left(M_n(t)\right)$ which is computed in the appendix. Its diverging behaviour is proven in Lemma A.3.

Lemma 4.5. *(i) If* $\sum \frac{[a_n t]}{2^n} < \infty$ *then there exists* $\Omega^\tau_M \subset \Omega^\tau$ *with* $\mathbb{P}\left(\Omega^\tau_M\right) = 1$ *such that on* Ω^τ_M *the following holds for all* $t > 0$

$$\lim_{n\to\infty} \left|\mathcal{E}\left(M_n(t)\right) - \mathbb{E}\mathcal{E}\left(M_n(t)\right)\right| = 0 \tag{4.54}$$

(ii) If $\sum \frac{[a_n t]}{2^n} = \infty$ *there exists* $\Omega^\tau_{M,n} \subset \Omega^\tau$ *with* $\mathbb{P}\left(\left(\Omega^\tau_{M,n}\right)^c\right) = 1 - o(1)$ *such that for* n *large enough on* $\Omega^\tau_{M,n}$ *the following holds for all* $t > 0$*:*

$$\left|\mathcal{E}\left(M_n(t)\right) - \mathbb{E}\mathcal{E}\left(M_n(t)\right)\right| \leq \left(\frac{[a_n t]}{2^n}\right)^{1/4} \tag{4.55}$$

Proof. We use a second order Tchebychev inequality to get $\forall\, \epsilon > 0$

$$\begin{aligned} & \mathbb{P}\left(\left|\mathcal{E}\left(M_n(t)\right) - \mathbb{E}\mathcal{E}\left(M_n(t)\right)\right| > \epsilon\right) \\ \leq\ & \epsilon^{-2}\left(\mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right)^2 - \left(\mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right)\right)^2\right) \end{aligned} \tag{4.56}$$

Calculating $\mathbb{E}\left(\mathcal{E}M_n(t)\right)^2$ yields

$$\mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right)^2 \equiv (d) + (e), \tag{4.57}$$

where

$$(d) \ = \ \mathbb{E}\left(\left(\frac{[a_n t]}{2^n}\right)^2 \sum_{x\in\mathcal{V}_n} g(\gamma_n(x))^2\right) \leq c_2 \frac{[a_n t]}{2^n}, \tag{4.58}$$

by the calculations in (A.12) and

$$(e) \ = \ \mathbb{E}\Bigg(\left(\frac{[a_n t]}{2^n}\right)^2 \sum_{x\in\mathcal{V}_n} \sum_{\substack{y\in\mathcal{V}_n,\\ y\neq x}} g(\gamma_n(x))g(\gamma_n(y))\Bigg). \tag{4.59}$$

From our calculations in (A.12) we know that $(d) \leq c_2 \frac{[a_n t]}{2^n}$. Moreover we have that

$$(e) - \left(\mathbb{E}\left(\mathcal{E}M_n(t)\right)\right)^2 = -\frac{[a_n t]^2}{2^n}\left(\mathbb{E}g(\gamma_n(x))\right)^2 < 0. \tag{4.60}$$

Thus we get

$$\mathbb{P}\left(\left|\mathcal{E}\left(M_n(t)\right) - \mathbb{E}\mathcal{E}\left(M_n(t)\right)\right| > \epsilon\right) \leq \epsilon^{-2} c_2 \frac{[a_n t]}{2^n} \tag{4.61}$$

In case (i) the bound in (4.61) is summable and thus we get the desired result using Borel Cantelli. In case (ii) we arrive directly at the desired expression. as desired. $\square$

5. Proof of Theorem 1.2

In this section we verify all remaining conditions of Theorem 2.1 , namely Condition (A3'). We then conclude the proof of Theorem 1.2.

5.1. **Verification of Condition (A3').** We turn to the verification of Condition (A3'). We define

$$\widehat{\lambda}_{\delta,n} \equiv \frac{a_n}{2^n} \sum_{x \in \mathcal{V}_n} f_\delta(\gamma_n(x)) \tag{5.1}$$

and observe that the quantity in (2.5) is $k_n(t)/a_n \widehat{\lambda}_{\delta,n}$.

Lemma 5.1. *Let c_n be an intermediate space scale and take $\beta = \beta_c(\varepsilon)$ with $0 < \varepsilon \leq 1$.*
(i) If $\sum_n \frac{a_n}{2^n} < \infty$ then there exists $\Omega^\tau_{10} \subset \Omega^\tau$ with $\mathbb{P}\left(\Omega^\tau_{10}\right) = 1$ such that on Ω^τ_{10} the following holds for all $t > 0$,

$$\lim_{\delta \to 0} \lim_{n \to \infty} \widehat{\lambda}_{\delta,n} = 0\,. \tag{5.2}$$

(ii) If $\sum_n \frac{a_n}{2^n} = \infty$ then there exists $\Omega^\tau_{n,11} \subset \Omega^\tau$ with $\mathbb{P}\left(\Omega^\tau_{n,11}\right) \geq 1 - o(1)$ such that for n large enough on $\Omega^\tau_{n,11}$ the following holds for all $t > 0$:

$$\left|\widehat{\lambda}_{\delta,n} - \mathbb{E}\left(\widehat{\lambda}_{\delta,n}\right)\right| \leq \left(\frac{a_n}{2^n}\right)^{1/4}, \tag{5.3}$$

and

$$\lim_{\delta \to 0} \lim_{n \to \infty} \mathbb{E}\left(\widehat{\lambda}_{\delta,n}\right) = 0\,. \tag{5.4}$$

Proof. We apply a second order Tchebychev inequality:

$$\mathbb{P}\left(\left|\widehat{\lambda}_{\delta,n} - \mathbb{E}\left(\widehat{\lambda}_{\delta,n}\right)\right| > \epsilon\right) \leq \epsilon^{-2}\left(\mathbb{E}\left(\widehat{\lambda}_{\delta,n}\right)^2 - \left(\mathbb{E}\left(\widehat{\lambda}_{\delta,n}\right)\right)^2\right) \tag{5.5}$$

Using that $f_\delta(\gamma_n(x))$ and $f_\delta(\gamma_n(x'))$ are independent if $x \neq x'$ we can bound (5.5) by

$$\epsilon^{-2} \tfrac{a_n}{2^n} a_n \mathbb{E}\left(f_\delta(\gamma_n(x))^2\right) \leq \epsilon^{-2} c_1 \tfrac{a_n}{2^n} \tag{5.6}$$

where we used (A.2). Again we are able to argument in the same way as in the proof of Proposition 6.4 in [9] which gives us the result of Proposition 5.1. □

5.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. Conditions (A1) and (A2) are satisfied $\mathbb{P}$-almost surely if $\frac{2^{m_n}}{2^n} \log n = o(1)$ and in $\mathbb{P}$-probability otherwise by the same argumentation as in the case $0 < \alpha(\epsilon) < 1$ given in [9], see the proof of Theorem 1.5 therein. by Lemma 5.1 Condition (A3') is satisfied $\mathbb{P}$-almost surely if $\frac{2^{m_n}}{2^n} \log n = o(1)$ and in $\mathbb{P}$-probability otherwise . Thus Theorem 2.1 (ii) implies that, w.r.t. the same convergence mode as (A1) holds, $S_n(\cdot) - \frac{M_n(\cdot)}{c_n} \Rightarrow S^{crit}(\cdot)$.

For the second part of Theorem 1.2, namely Eq. (1.14) we proceed as follows. Theorem 4.1 ensure the finite dimensional convergence in law of $M_n(\cdot) - \mathbb{E}(\mathcal{E}(M_n(1)))(\cdot)$ to 0 on $[0, T]$, either $\mathbb{P}$-a. s. or in $\mathbb{P}$-probability. Since $M_n(t)$ is an increasing process with a continuous limit tightness follows by Theorem VI.3.37 in [11]. Hence we get the weak convergence of$M_n(\cdot) - \mathbb{E}(\mathcal{E}(M_n(1)))(\cdot)$ which is equivalent to convergence in probability since the limiting object is constant. This convergence also holds either $\mathbb{P}$-a. s. or in $\mathbb{P}$-probability. This yields the result of Theorem 1.2. □

6. Consequences for correlation functions (intermediate scales).

Lemma 6.1. *Let $\beta = \beta_c(\varepsilon)$ with $0 < \varepsilon \leq 1$. Let c_n be an intermediate scale with $\lim_{n\to\infty} \frac{\log c_n}{\beta\sqrt{n}} - \beta\sqrt{n} = \theta$ for some constant $\theta \in (-\infty, \infty)$. If $\sum a_n/2^n < \infty$ we have for all $T > 0$ and for all $\epsilon > 0$*

$$\lim_{n\to\infty} \mathcal{P}\left(\sup_{t\in[0,T]} \left| \frac{S_n(t)}{\mathbb{E}(\mathcal{E}(M_n(1)))} - t \right| > \epsilon \right) = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.1}$$

If $\sum a_n/2^n = \infty$ the same holds in $\mathbb{P}$-probability.

Proof. The claim of Lemma 6.1 follows immediately from Theorem 1.2. □

We define $\widetilde{a}_n$ by

$$\widetilde{a}_n e^{n\beta^2/2}\Phi(\theta) = c_n, \tag{6.2}$$

where Φ is the normal distribution function and $\theta = \lim_{n\to\infty} \frac{\log c_n}{\sqrt{n}\beta} - \sqrt{n}\beta$. A crucial quantity is the ratio $\widetilde{a}_n/a_n$ which we can compute explicitly using (3.4) and (6.2)

$$\frac{\widetilde{a}_n}{a_n} = \frac{e^{-\theta^2/2}}{\Phi(\theta)\beta\sqrt{2\pi n}}(1 + o(1)). \tag{6.3}$$

Set

$$A_n(t) \equiv \mathcal{P}\left(\sup_{k\in\{1,\dots,\lfloor\widetilde{a}_n t\rfloor\}} \left| \widetilde{S}_n(k) - \frac{c_n k}{\widetilde{a}_n} \right| > c_n\epsilon \right). \tag{6.4}$$

We need the following preperatory lemma.

Lemma 6.2. *If $\sum a_n/2^n < \infty$ we have for all $t, s > 0$ and for all $\epsilon > 0$*

$$\lim_{n\to\infty} \sqrt{n}\mathcal{P}\left(\left| \sum_{k=1}^{\lfloor\widetilde{a}_n t\rfloor} \mathcal{P}(\tau_n(J_n(k+1))e_{n,k+1} > c_n s | J_n(k)) - \frac{\widetilde{a}_n t}{a_n s} \right| > \frac{\widetilde{a}_n \epsilon}{a_n\sqrt{A_n}} \right) = 0 \quad \mathbb{P}\text{-a.s.} \tag{6.5}$$

If $\sum a_n/2^n = \infty$ the same holds in $\mathbb{P}$-probability.

Proof. Proceeding as in the verification of Condition (A1) we obtain analogue to Proposition 4.1 in [9]

$$\begin{aligned}
&\mathcal{P}\left(\left| \sum_{k=1}^{\lfloor\widetilde{a}_n t\rfloor} \mathcal{P}(\tau_n(J_n(k+1))e_{n,k+1} > c_n s | J_n(k)) - \frac{\lfloor\widetilde{a}_n t\rfloor}{a_n}\nu_n(s,\infty) \right| > \frac{\widetilde{a}_n \epsilon}{a_n\sqrt{A_n}} \right) \\
&\leq \left(\frac{a_n\sqrt{A_n(t)}}{\widetilde{a}_n \epsilon} \right)^2 \widetilde{\Theta}_n(s,t),
\end{aligned} \tag{6.6}$$

where

$$\widetilde{\Theta}_n(s,t) = \left(\frac{\lfloor\widetilde{a}_n t\rfloor}{a_n}\right)^2 \frac{\nu_n^2(u,\infty)}{2^n} + \frac{\lfloor\widetilde{a}_n t\rfloor}{a_n}\left(\sigma_n^2(u,\infty) + c\frac{\nu_n(2u,\infty)}{n^2} + \rho_n \left[\mathbb{E}\nu_n(u,\infty)\right]^2\right). \tag{6.7}$$

Using Proposition 5.1 in [9] and (6.3) we obtain the claim of Lemma 6.2. □

Proof of Theorem 1.3. Fix a realization of the random environment such that for all $t, T > 0$, for all $x > s$ uniformly in x and for all $\epsilon > 0$

$$\sqrt{n}\mathcal{P}\left(\left|\sum_{k=1}^{\lfloor \widetilde{a}_n t\rfloor} \mathcal{P}(\tau_n(J_n(k+1))e_{n,k+1} > c_n s | J_n(k)) - \frac{\widetilde{a}_n t}{a_n s}\right| > \frac{\widetilde{a}_n \epsilon}{a_n\sqrt{A_n(t)}}\right) = 0 \quad (6.8)$$

$$\lim_{n\to\infty} A_n(t) = 0. \quad (6.9)$$

$$\lim_{n\to\infty}\left|\lfloor a_n t\rfloor \mathcal{P}\left(\tau_n(J_n(k+1)e_{n,k+1} > c_n x\right) - \frac{t}{x}\right| = 0. \quad (6.10)$$

and finally,

$$\lim_{n\to\infty}\sqrt{n}\mathcal{P}\left(\widetilde{M_n}(\lfloor \widetilde{a}_n t(1+\epsilon')/a_n\rfloor) < c_n\right) = 0 \quad (6.11)$$

Rewriting the correlation function gives

$$\begin{aligned}
&\mathcal{C}_n(t_n, s_n) \\
= \;& \mathcal{P}\left(\bigcup_{k>0} \widetilde{S}_n(k) < t_n, \widetilde{S}_n(k+1) > t_n + s_n\right) \\
= \;& \mathcal{P}\left(\bigcup_{k\leq \widetilde{a}_n(1+\epsilon')} \widetilde{S}_n(k) < t_n, \tau_n(J_n(k+1))e_{n,k+1} > t_n + s_n - \widetilde{S}_n(k)\right) \\
& + \mathcal{P}\left(\bigcup_{k> \widetilde{a}_n(1+\epsilon')} \widetilde{S}_n(k) < t_n, \tau_n(J_n(k+1))e_{n,k+1} > t_n + s_n - \widetilde{S}_n(k)\right) \quad (6.12)
\end{aligned}$$

Using that $\widetilde{S}_n(k)$ is an increasing process and (6.11), we have for all $\epsilon > 0$

$$\mathcal{P}\left(\bigcup_{k>\widetilde{a}_n(1+\epsilon')}\left\{\widetilde{S}_n(k) \leq t_n\right\}\right) \leq \mathcal{P}\left(\frac{\widetilde{S}_n(\lfloor \widetilde{a}_n t(1+\epsilon')\rfloor)}{c_n} < t\right). \quad (6.13)$$

By only counting summands smaller than δ and we bound (6.13) from above by

$$\mathcal{E}\left(M_n(\widetilde{a}_n/a_n t(1+\epsilon'))\right). \quad (6.14)$$

Hence we can rewrite (6.12) as

$$\begin{aligned}
& \sum_{k\in T}\mathcal{P}\left(\widetilde{S}_n(k) < t_n, \tau_n(J_n(k+1)e_{n,k+1} > t_n + s_n - \widetilde{S}_n(k)\right) + o(1/\sqrt{n}), \quad (6.15) \\
= \;& \sum_{k\in T}\left\{\mathcal{E}\left(\mathbb{1}_{\{\widetilde{S}_n(k)<t_n\}}\mathbb{1}_{\left\{\left|\widetilde{S}_n(k)-\frac{c_n k}{\widetilde{a}_n}\right|>c_n\epsilon\right\}}\mathbb{1}_{\{\tau_n(J_n(k+1)e_{n,k+1}>t_n+s_n-\widetilde{S}_n(k)\}}\right)\right. \\
+ \;& \left.\mathcal{E}\left(\mathbb{1}_{\{\widetilde{S}_n(k)<c_n t\}}\mathbb{1}_{\left\{\left|\widetilde{S}_n(k)-\frac{c_n k}{\widetilde{a}_n}\right|\leq c_n\epsilon\right\}}\mathbb{1}_{\{\tau_n(J_n(k+1)e_{n,k+1}>t_n+s_n-\widetilde{S}_n(k)\}}\right)\right\} + o(1/\sqrt{n}),
\end{aligned}$$

where $T = \{0, \ldots, \lfloor \widetilde{a}_n(1+\epsilon)\rfloor\}$. We want to show that the summands including the event $\left\{\left|\widetilde{S}_n(k) - \frac{c_n k}{\widetilde{a}_n}\right| > c_n\epsilon\right\}$ are not contributing in the limit namely that they are of order

$o(1/\sqrt{n})$. Using the following bound from above we have

$$\begin{aligned}
& \sum_{k\in T}\mathcal{E}\left(\mathbb{1}_{\{\widetilde{S}_n(k)<t_n\}}\mathbb{1}_{\left\{\left|\widetilde{S}_n(k)-\frac{c_nk}{\widetilde{a}_n}\right|>c_n\epsilon\right\}}\mathbb{1}_{\{\tau_n(J_n(k+1)e_{n,k+1}>t_n+s_n-\widetilde{S}_n(k))\}}\right) \\
\leq\ & \sum_{k\in T}\mathcal{E}\left(\mathbb{1}_{\left\{\left|\widetilde{S}_n(k)-\frac{c_nk}{\widetilde{a}_n}\right|>c_n\epsilon\right\}}\mathbb{1}_{\{\tau_n(J_n(k+1)e_{n,k+1}>t_n+s_n-\widetilde{S}_n(k))\}}\right) \\
\leq\ & \mathcal{E}\left(\mathbb{1}_{\left\{\sup_{k\in[0,\widetilde{a}_n(1+\epsilon)]\cap\mathbb{N}}\left|\widetilde{S}_n(k)-\frac{c_nk}{\widetilde{a}_n}\right|>c_n\epsilon\right\}}\sum_{k\in T}\mathcal{P}\left(\tau_n(J_n(k+1))e_{n,k+1}>s_n|J_n(k)\right)\right).
\end{aligned} \tag{6.16}$$

Using (6.8) we bound (6.16) by

$$\begin{aligned}
& \frac{\widetilde{a}_n}{a_n}\left(\frac{1}{s}+\frac{1}{\sqrt{A_n}}\epsilon\right)\mathcal{P}\left(\sup_{k\in[0,\ldots,\widetilde{a}_n(1+\epsilon)]}\left|\widetilde{S}_n(k)-\frac{c_nk}{\widetilde{a}_n}\right|>c_n\epsilon\right)+o(1/\sqrt{n}) \\
=\ & \frac{\widetilde{a}_n}{a_n}\frac{1}{s}A_n(1+\epsilon)+\frac{\widetilde{a}_n\sqrt{A_n(1+\epsilon)}}{a_n}\epsilon+o(1/\sqrt{n}).
\end{aligned} \tag{6.17}$$

By (6.9) we have that (6.17) is of order $o(\widetilde{a}_n/a_n)=o(1/\sqrt{n})$. We can bound the second summand in (6.15) from above by

$$\begin{aligned}
& \sum_{k\in[0,\ldots,\widetilde{a}_n(1-\epsilon)]}\mathcal{P}\left(\tau_n(J_n(k+1)e_{n,k+1}>t_n+s_n-c_n\left(\frac{k}{\widetilde{a}_n}+\epsilon\right)\right) \\
& \quad +2\epsilon\widetilde{a}_n\mathcal{P}\left(\tau_n(J_n(k+1)e_{n,k+1}>s_n\right)
\end{aligned} \tag{6.18}$$

By (6.10) (6.18) is equal to

$$\sum_{k\in[0,\ldots,\widetilde{a}_n(1-\epsilon)]}\frac{1}{a_n}\left(1+s-\left(\frac{k}{\widetilde{a}_n}+\epsilon\right)\right)^{-1}+2\epsilon\frac{\widetilde{a}_n}{a_n}+\frac{\widetilde{a}_n}{a_n}o(1) \tag{6.19}$$

In the same way we obtain the following lower bound on (6.15):

$$\sum_{k\in[0,\ldots,\widetilde{a}_n(1+\epsilon)]}\frac{1}{a_n}\left(1+s-\left(\frac{k}{\widetilde{a}_n}-\epsilon\right)\right)^{-1}-2\epsilon\frac{\widetilde{a}_n}{a_n}-\frac{\widetilde{a}_n}{a_n}o(1) \tag{6.20}$$

By a Riemann sum argument we have

$$\begin{aligned}
\sum_{k\in\{0,\ldots,\widetilde{a}_n(1-\epsilon)\}}\frac{1}{\widetilde{a}_n}\left(1+s-\left(\frac{k}{\widetilde{a}_n}+\epsilon\right)\right)^{-1}-\frac{1}{\widetilde{a}_n}\frac{1}{1+s} &\geq \int_0^{1-\epsilon}\left(1+s-(t+\epsilon)\right)^{-1}\mathrm{d}t \\
&= \log\left(\frac{s+1-\epsilon}{s}\right),
\end{aligned} \tag{6.21}$$

respectively

$$\begin{aligned}
\sum_{k\in\{0,\ldots,\widetilde{a}_n(1+\epsilon)\}}\frac{1}{\widetilde{a}_n}\left(1+s-\left(\frac{k}{\widetilde{a}_n}-\epsilon\right)\right)^{-1}-\frac{1}{\widetilde{a}_n}\frac{1}{s} &\leq \int_0^{1+\epsilon}\left(1+s-(t-\epsilon)\right)^{-1}\mathrm{d}t \\
&= \log\left(\frac{1+s+\epsilon}{s}\right)
\end{aligned} \tag{6.22}$$

Hence we can bound (6.19) from above by

$$\frac{\widetilde{a}_n}{a_n}\log\left(\frac{s+1-\epsilon}{s}\right)-\frac{1}{\widetilde{a}_n}\frac{1}{1+s}+2\epsilon\frac{\widetilde{a}_n}{a_n}+\frac{\widetilde{a}_n}{a_n}o(1) \tag{6.23}$$

and (6.20) from below by

$$\frac{\widetilde{a}_n}{a_n}\log\left(\frac{1+s+\epsilon}{s}\right)-\frac{1}{\widetilde{a}_n}\frac{1}{s}-2\epsilon\frac{\widetilde{a}_n}{a_n}-\frac{\widetilde{a}_n}{a_n}o(1) \tag{6.24}$$

Putting those estimates together we obtain

$$\mathcal{C}_n(t_n,t_n+s_n)=\frac{\widetilde{a}_n}{a_n}\log\left(1+\frac{1}{s}\right)+\frac{\widetilde{a}_n}{a_n}o(1). \tag{6.25}$$

By (6.2) we have

$$\mathcal{C}_n(t_n,t_n+s_n)=\frac{1}{\Phi(\theta)}e^{-\theta^2/2}\log\left(1+\frac{1}{s}\right)\frac{1}{\beta\sqrt{2\pi n}}(1+o(1)). \tag{6.26}$$

So far we worked on a fixed realization of the random environment. Observe that (6.10) can be rewritten as

$$\frac{\lfloor\widetilde{a}_n t\rfloor}{a_n}\nu_n(x,\infty). \tag{6.27}$$

Hence (6.10) holds either P-a.s. or in $\mathbb{P}$-probability by Proposition 5.1 in [9] and the observation that $\nu_n(x,\infty)$ is a monotone function in x and its limit $1/x$ is continuous for $x>s$. By Lemma 6.1 a nd6.2 (6.8) and (6.9) hold either $\mathbb{P}$-a.s. or in $\mathbb{P}$-probability. (6.11) holds either $\mathbb{P}$-a.s. or in $\mathbb{P}$-probabilty because Codition (B1) holds $\mathbb{P}$-a.s or in $\mathbb{P}$-probability. Arguing as in the proof of Theorem 1.3 in [10] we have that if (6.8),(6.9), (6.10) and (6.11) hold $\mathbb{P}$-a.s, respectively in $\mathbb{P}$-probability, (6.26) holds with respect to the same convergence mode with respect to the random environment. This concludes the proof of Theorem 1.3. □

Appendix A. Calculations

In the appendix we give results on the moments of $f_\delta(\gamma_n(x))$ and $g_\delta(\gamma_n(x))$ which were needed before. During the verification of Condition (A3') we needed to compute the asymptotic behavior of $a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\right)$ and $a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\right)$. This we want to state in the following Lemma.

Lemma A.1. *For all $t>0$ and $\delta>0$ and for n large enough there exist constants $0<c_0,c_1<\infty$ such that*

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\right)\leq c_0\delta \tag{A.1}$$

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\right)\leq\delta^4+c_1. \tag{A.2}$$

Proof. We observe that

$$f_\delta(u)\leq\delta^2\quad\forall u\in(0,\infty). \tag{A.3}$$

We decompose $a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\right)$ in the following way

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\right)=a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\mathbb{1}_{\{\gamma_n(x)>\delta\}}\right)+a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\mathbb{1}_{\{\gamma_n(x)\leq\delta\}}\right)\equiv(1)+(2) \tag{A.4}$$

We start by controlling the behavior of (1).

$$(1)\leq\delta^2a_n\mathbb{P}\left(\gamma_n(x)>1\right)\sim\delta, \tag{A.5}$$

where we used the definition of a_n and c_n. Turning to (2) we have

$$\begin{aligned}(2) &\leq a_n\mathbb{E}\left(\gamma_n(x)^2\mathbb{1}_{\{\gamma_n(x)\leq\delta\}}\right)\\ &= \frac{a_n e^{2n\beta^2}}{c_n^2}\int_{-\infty}^{\frac{\log(c_n\delta)}{\sqrt{n}\beta}-2\sqrt{n}\beta}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u\\ &\sim \frac{a_n e^{2n\beta^2}}{c_n^2}\left(\sqrt{2\pi}\left(-\frac{\log(c_n\delta)}{\sqrt{n}\beta}+2\sqrt{n}\beta\right)\right)^{-1}e^{-\frac{1}{2}\left(\frac{\log(c_n\delta)}{\sqrt{n}\beta}-2\sqrt{n}\beta\right)^2},\end{aligned}\tag{A.6}$$

where we used that by (2.21) in [9] $\frac{\log(c_n\delta)}{\sqrt{n}\beta}-2\sqrt{n}\beta\to-\infty$ as $n\to\infty$. Using (2.21) in [9] to expand we have that (A.6) is equal to

$$a_n\left(\sqrt{2\pi}\left(-\frac{\log(c_n\delta)}{\sqrt{n}\beta}+2\sqrt{n}\beta\right)\right)^{-1}\delta^2 e^{-\frac{1}{2}\left(\frac{\log c_n\delta}{\sqrt{n}\beta}\right)^2}=c_0'\delta(1+o(1)),\tag{A.7}$$

where $0<c_0'<\infty$. Putting our estimates together we have that for n large enough there exists a constant $0<c_0<\infty$ such that

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))\right)\leq c_0\delta.\tag{A.8}$$

In a similar way we treat $a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\right)$. This time we truncate at one, namely

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\right)=a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\mathbb{1}_{\{\gamma_n(x)>1\}}\right)+a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\mathbb{1}_{\{\gamma_n(x)\leq1\}}\right).\tag{A.9}$$

For the first summand we use again the bound on f and the definition of the timescale to bound it by δ^4. And for the second summand we use the same method as for (2): Applying Gaussian estimates, expanding the resulting term and plugging in the exact representation of c_n. The bound we obtain is a constant. Putting these estimates together we have for n large enough

$$a_n\mathbb{E}\left(f_\delta(\gamma_n(x))^2\right)\leq\delta^4+c_1.\tag{A.10}$$

□

To study the behavior of $M_n(t)$ in particular to show Condition (B1) we needed a control on the moments of $g_1(\gamma_n(x))$ when $\beta=\beta_c$ which is done in the following Lemma.

Lemma A.2. *Let c_n be an intermediate scale.*
(i) If $\lim_{n\to\infty}\frac{\log(c_n)}{\beta\sqrt{n}}-\beta\sqrt{n}=c$ for some $c\in(-\infty,\infty)$. Then

$$a_n\mathbb{E}\left(g(\gamma_n(x))\right)=\Phi(c)\frac{a_n e^{n\beta^2/2}}{c_n}(1+o(1)).\tag{A.11}$$

(ii) For n large enough there exists a constant $0<c_2<\infty$ such that

$$a_n\mathbb{E}\left(g(\gamma_n(x))^l\right)\leq c_2\quad 2\leq l\leq4.\tag{A.12}$$

Proof. Recall that $g(u)\leq1\quad\forall u>0$. First we proof assertion (i). We rewrite $a_n\mathbb{E}\left(g(\gamma_n(x))\right)$ as

$$\begin{aligned}&\frac{a_n e^{n\beta^2/2}}{\sqrt{2\pi}c_n}\int_{-\infty}^{\infty}e^{\sqrt{n}\beta z}\left(1-e^{-c_n e^{-\sqrt{n}\beta z}}\right)e^{-z^2/2}\mathrm{d}z\\ =\;&\frac{a_n}{c_n}e^{n\beta^2/2}-\frac{[a_nt]e^{n\beta^2/2}}{c_n\beta\sqrt{2\pi n}}\int_{-\infty}^{\infty}e^{y+\log(c_n)}e^{-\left(\frac{y}{\beta\sqrt{n}}+\frac{\log c_n}{\beta\sqrt{n}}\right)^2-e^y}\mathrm{d}y\end{aligned}\tag{A.13}$$

Now one can cut the domain of integration into different pieces. Observe that in the region $y < -\log n$ the integral is equal to

$$\begin{aligned} & (1+o(1))\frac{a_n e^{n\beta^2/2}}{c_n\beta\sqrt{2\pi n}}\int_{-\infty}^{-\log n} e^{y+\log(c_n)}e^{-\left(\frac{y}{\beta\sqrt{n}}+\frac{\log c_n}{\beta\sqrt{n}}\right)^2}\mathrm{d}y \\ = \; & (1+o(1))\frac{a_n e^{n\beta^2/2}}{c_n\sqrt{2\pi}}\int_{-\infty}^{-\frac{\log n}{\sqrt{n}\beta}-\frac{\log c_n}{\beta\sqrt{n}}+\sqrt{n}\beta} e^{-y^2/2}\mathrm{d}y \end{aligned} \tag{A.14}$$

We now distingish several cases. If $\frac{\log c_n}{\beta\sqrt{n}} - \sqrt{n}\beta \to c$ for some constant c as $n \to \infty$ we have that (A.13) is equal to $(1+o(1))\frac{a_n e^{n\beta^2/2}}{c_n}\Phi(c)$. If $\frac{-\log c_n}{\beta\sqrt{n}} + \sqrt{n}\beta \to \infty$ or $-\infty$ one uses Gaussian estimates to obtain the exact asymptotic behavior. One can bound the integral on the other part of the domain of integration to see that it is bounded by a constant.. This yields (A.11). We now turn to assertion (ii) and consider $\mathbb{E}\left(g(\gamma_n(x))^2\right)$. We will split this into two terms:

$$a_n\mathbb{E}\left(g(\gamma_n(x))^2\right) = a_n\mathbb{E}\left(g(\gamma_n(x))^2\mathbb{1}_{\{\gamma_n(x)>1\}}\right) + a_n\mathbb{E}\left(g(\gamma_n(x))^2\mathbb{1}_{\{\gamma_n(x)\le 1\}}\right) \equiv (1)+(2). \tag{A.15}$$

For (1) we use the definition of the scaling a_n and c_n and the bound $g(u) \le 1 \quad \forall u > 0$.

$$(1) \le a_n\mathbb{P}\left(\gamma_n(x) > 1\right) = 1. \tag{A.16}$$

For Term (2) we use exact Gaussian estimates

$$\begin{aligned} (2) \;\le\; & \frac{a_n}{c_n^2}\int_{-\infty}^{\frac{\log(c_n)}{\sqrt{n}\beta}} e^{2\beta\sqrt{n}\beta u}\left(1-e^{-c_n e^{-\sqrt{n}\beta u}}\right)\frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u \\ \le\; & \frac{a_n e^{2n\beta^2}}{c_n^2}\int_{-\infty}^{\frac{\log(c_n)}{\sqrt{n}\beta}-2\sqrt{n}\beta}\frac{e^{-r^2/2}}{\sqrt{2\pi}}\mathrm{d}r \\ \sim\; & \frac{a_n e^{2n\beta^2}}{c_n^2}\left(\sqrt{2\pi}\left(-\frac{\log(c_n)}{\sqrt{n}\beta}+2\sqrt{n}\beta\right)\right)^{-1} e^{-\left(\frac{\log(c_n)}{\sqrt{n}\beta}-2\sqrt{n}\beta\right)^2/2}, \end{aligned} \tag{A.17}$$

where we use that by (2.22) in [9] we have $\frac{\log(c_n)}{\sqrt{n}\beta} - 2\sqrt{n}\beta \to -\infty$ as $n \to \infty$. We plug (2.22) in [9] into (A.17) and obtain that (A.17) is equal to

$$a_n\left(\sqrt{2\pi}\left(-\frac{\log(c_n)}{\sqrt{n}\beta}+2\sqrt{n}\beta\right)\right)^{-1} e^{-\left(\frac{\log c_n}{\sqrt{n}\beta}\right)^2/2} = c_2'(1+o(1)), \tag{A.18}$$

where $0 < c_2' < \infty$. Putting both estimates together we get that for n large there exists a constant $0 < c_2 < \infty$ such that

$$a_n\mathbb{E}\left(g_1(\gamma_n(x))^2\right) \le c_2. \tag{A.19}$$

In exactly the same way one proceeds with $a_n\mathbb{E}\left(g(\gamma_n(x))^3\right)$ and $a_n\mathbb{E}\left(g(\gamma_n(x))^4\right)$ to obtain (A.12) for $l = 3$ and $l = 4$. $\square$

Computing $\mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right)$ at $\beta=\beta_c(\epsilon)$ gives

$$
\begin{aligned}
\mathbb{E}\left(\mathcal{E}\left(M_n(t)\right)\right) &= \mathbb{E}\left(\sum_{i=1}^{[a_n t]}\mathcal{E}\left(\frac{\tau_n(J_n(i))e_{n,i}}{c_n}1_{\left\{0<\frac{\tau_n(J_n(i))e_{n,i}}{c_n}<\delta\right\}}\right)\right)\\
&= \frac{[a_n t]}{\sqrt{2\pi}c_n}\int_{-\infty}^{\infty}e^{\sqrt{n}\beta z}\left(1-e^{-c_n\delta e^{-\sqrt{n}\beta z}}\right)e^{-z^2/2}\mathrm{d}z\\
&= \frac{[a_n t]}{c_n}e^{n\beta^2/2}-\frac{[a_n t]}{c_n\beta\sqrt{2\pi n}}\int_{-\infty}^{\infty}e^{-\frac{y^2}{2\beta^2 n}-\delta e^{y}}\mathrm{d}y\\
&= (1-o(1))c\frac{[a_n t]}{c_n}e^{n\beta^2/2}, \qquad (A.20)
\end{aligned}
$$

where c is some constant >0.

The following lemma proves the general diverging behavior of $\mathbb{E}(\mathcal{E}(M_n(1))$

Lemma A.3. *Let $\beta=\beta_c$ and a_n and c_n scalings with*

$$a_n\mathbb{P}(\tau_n(\sigma)>c_n)=1. \qquad (A.21)$$

Then

$$\lim_{n\to\infty}\frac{a_n e^{n\beta^2/2}}{c_n}=\infty. \qquad (A.22)$$

Proof. We use that $\log(a_n)=\frac{n\beta_c^2}{2}(1+o(1))$. Hence there exists some sequence $f(n)$ with $\frac{f(n)}{n\beta_c^2}=o(1)$ with $\log(a_n)=\frac{n\beta_c^2+f(n)}{2}$. By definition of c_n we have

$$\frac{\log(c_n)}{\sqrt{n}\beta}=\sqrt{2\log(a_n)}-\frac{1/2\log(\log(a_n))+\log(4\pi)}{\sqrt{2\log(a_n)}}. \qquad (A.23)$$

We have $\log(\log(a_n))=\log(\frac{n\beta_c^2+f(n)}{2})$ and $\sqrt{2log(a_n)}=\sqrt{n\beta^2+f(n)}$. Observe that do to the asymptotic behavior of $f(n)$ $\log(\log(a_n))$ is positive fpr n large enough. Hence it suffices to show

$$\frac{\log(a_n e^{n\beta^2/2})}{\sqrt{n}\beta}\le\sqrt{2\log a_n} \qquad (A.24)$$

Plugging in the expressions for $\log(a_n)$ (A.24) reads

$$\sqrt{n}\beta+\frac{f(n)}{2\sqrt{n}\beta}\le\sqrt{n\beta^2+f(n)}, \qquad (A.25)$$

which is always satisfied and equality holds if and only if $f(n)=0$. □

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